13/06/2018

Langlands p-adic

Lecture 1: Cohomology at level 0.

Goal: . explain some new phenomena arising when dealing with p-adic étale coho. of "big spaces"

. Use these phenomena / computations to get a geo. realisation of p-adic LL for $GL_2(\mathbb{Q}_p)$ (in many cases).

* Etale vs pro-étale p-adic coho.

$C/\mathbb{Q}_p$ complete alg. closed

X/C qcqs rigid space $\rightarrow H^*_{et}(X,\mathbb{Q}_p) = H^*_{proet}(X,\mathbb{Q}_p)$ (Scholze)

proper:

$H^*_{et}(X,\mathbb{Q}_p)\otimes_{\mathbb{Q}_p} C \xrightarrow{\sim} H^*_{proet}(X,\widehat{\mathcal{O}})$.

⚠. All these fail very badly outside this context.

. $H^*_{et} \rightarrow H^*_{proet}$ is not even injective in general.

$X = \bigcup_{n\geq 1} U_n$ Stein covering.

$$H^1_{et}(X,\mathbb{Q}_p) = (\varprojlim H^1_{et}(U_n,\mathbb{Z}_p))\left[\tfrac{1}{p}\right]$$

$$H^1_{proet}(X,\mathbb{Q}_p) = \varprojlim \left(H^1_{et}(U_n,\mathbb{Z}_p)\left[\tfrac{1}{p}\right]\right).$$

Big pbm: Rlim everywhere.

$\mathbb{A}^n$ affine space

Thm 1) (Berkovich) $H^*_{et}(\mathbb{A}^n, \mathbb{Z}/p^k) = 0 \quad \forall k$ $\quad * > 0$

$(\Rightarrow H^*_{et}(\mathbb{A}^n,\mathbb{Q}_p) = 0)$

2) (Colmez - Niziol, Le Bras)

$$H^i_{proet}(\mathbb{A}^n, \mathbb{Q}_p(i)) \xrightarrow{\sim} \underbrace{\Omega^i(\mathbb{A}^n)^{d=0}}_{\text{Huge } C\text{-vs}}$$

Rmk. Same result (2) holds for open polydiscs.

Drinfeld's space : $K/\mathbb{Q}_p$ finite

$$\Omega^d_K = \mathbb{P}^d_K \setminus \bigcup_{K\text{-ratio hyperplanes}} H$$

$\ell \neq p$ ℓ-adic cohomology computed by Schneider-Stuhler.

$G = GL_{d+1}(K) \curvearrowright \Omega^d_K$

$\{1, \dots, d\}$ = simple roots of G wrt upper

$\rightsquigarrow \left(\begin{array}{l} \mathcal{P}(\{1, \dots d\}) \hookrightarrow \text{std parabolic} \\ \qquad I \longrightarrow P_I \end{array}\right.$

$$St_I(A) = LC(G/P_I, A) \Big/ \sum_{J \supsetneq I} LC(G/P_J, A)$$

St_I^{cont} ...

Thm (S.S.) $\quad H^i_{et}(\Omega^d_K \hat{\otimes} C, \mathbb{Q}_\ell(i)) \xrightarrow{\sim} (St_i^{cont}(\mathbb{Q}_\ell))^*$

$0 \le i \le d.$ $\quad H^i_{proét}(\qquad\qquad) \xrightarrow{\sim} St_i(\mathbb{Q}_\ell)^*$

(where $St_i = St_{\{1, \dots, d-i\}}$).

Thm (CDN) 1) $H^i_{et}(\Omega^d_K \hat{\otimes} C, \mathbb{Q}_p(i)) \cong St_i^{cont}(\mathbb{Q}_p)^*$.

2) $$0 \to \frac{\Omega^{i-1}(\Omega^d_K \hat{\otimes} C)}{\operatorname{Ker} d} \to H^i_{proét}(\Omega^d_K \hat{\otimes} C, \mathbb{Q}_p(i)) \to St_i(\mathbb{Q}_p)^* \to 0.$$

- Case of dim 1 : applied p-adic analysis.

Lubin-Tate $\quad D = \{|z| < 1\}/_C$

Proposition. $H^1_{proét}(D, \mathbb{Q}_p(1)) \simeq \mathcal{O}(D)/_C$

$H^i_{proét}(D, \mathbb{Q}_p(1)) = 0 \qquad i > 1.$

Proof. $r_n \nearrow 1 \qquad D_n = \{|z| \le r_n\}$

$$H^1_{proét}(D, \mathbb{Q}_p(1)) = \varprojlim H^1_{et}(D_n, \mathbb{Q}_p(1))$$

Kummer seq $\rightsquigarrow H^1_{et}(D_n, \mathbb{Q}_p(1)) \xrightarrow{\sim} A_n \hat{\otimes} \mathbb{Q}_p = \widehat{A_n}\left[\frac{1}{p}\right]$

$(A_n := \mathcal{O}(D_n)^\times / C^\times)$

$H^1_{proét}(D, \mathbb{Q}_p(1)) \cong \varprojlim (A_n \hat{\otimes} \mathbb{Q}_p)$

$f \in \mathcal{O}(D_n)^\times \Rightarrow \left\| \frac{f}{f(0)} - 1 \right\| < 1 \Rightarrow \log \frac{f}{f(0)} \in \mathcal{O}(D_n)$

⚠ $f \mapsto \log \frac{f}{f(0)}$ does not extend to $\hat{A}_n$

does ~~not~~ extend to a map

$$A_n \hat{\otimes} \mathbb{Q}_p \xrightarrow[\log]{} \mathcal{O}(D_{n-1})/C$$

$\longrightarrow$ $\{A_n \hat{\otimes} \mathbb{Q}_p\} \underset{\exp}{\xrightarrow[\log]{\sim}} \{\mathcal{O}(D_n)/C\}$ as pro-systems.

$$\begin{tikzcd}
0 \arrow[r] & R^1\varprojlim H^n_{ét}(D_n, \mathbb{Q}_p(1)) \arrow[r] \arrow[d, equal] & H^2_{proét}(D, \mathbb{Q}_p(1)) \arrow[d] \\
& R^1\varprojlim (A_n \hat{\otimes} \mathbb{Q}_p) & \varprojlim H^2_{ét}(D_n, \mathbb{Q}_p(1)) \arrow[d] \\
& & 0
\end{tikzcd}$$

$\varprojlim H^2_{ét}(D_n, \mathbb{Q}_p(1)) = 0$ (Berkovich)

$= R^1\varprojlim (\mathcal{O}(D_n)/C)$

$= 0$ (Kiehl)

□.

<u>For étale cohomology</u>:

Kummer sequence:

$$0 \to \widehat{\mathcal{O}(D)^\times/C^\times} \longrightarrow H^1_{ét}(D, \mathbb{Z}_p(1)) \longrightarrow T_p(\mathrm{Pic}\, D) \to 0$$

$$\widehat{\mathcal{O}(D)^\times/C^\times} \cong 1 + T\mathcal{O}_C[[T]]$$

<u>Thm</u> (CDN). 1) $T_p(\mathrm{Pic}\, D) = 0$ ($(\mathrm{Pic}\, D)_{tors}$ has bounded exponent).

2) $H^2_{ét}(D, \mathbb{Z}_p(1)) = \widehat{\mathrm{Pic}\, D}$

and this is $0 \Leftrightarrow C$ spherically complete.

<u>Rmk</u>. $(\mathrm{Pic}\, D)[p] = 0 \Leftrightarrow C$ spherically complete.

<u>Thm</u> (Lazard) $\mathrm{Pic}\, D = 0 \Leftrightarrow C$ spherically complete.

<u>Proof of $\Leftarrow$</u>:

$$0 \to R^1\varprojlim \mathcal{O}(D_n)^\times \longrightarrow \mathrm{Pic}\, D \longrightarrow \underbrace{\varprojlim \mathrm{Pic}(D_n)}_{0} \to 0$$

Need: $\forall f_n \in \mathcal{O}(D_n)^\times \ \exists g_n \in \mathcal{O}(D_n)^\times$ s.t. $f_n = \frac{g_n}{g_{n+1}}$

$g_n = f_n f_{n+1} f_{n+2} f_{n+3} \cdots$

Any bounded sequence in C has a limit.

$$\begin{array}{ccc} \ell^\infty(C) \supset \{\text{convergent seq}\} & & (a_n) \\ \downarrow & & \downarrow \\ C & & \lim_{n\to+\infty} a_n \end{array}$$

Hahn-Banach $\Rightarrow \lim_{n\to+\infty}$ extends to a c^o lin. form of norm ≤ 1 $\left| \begin{array}{l} \ell^\infty(C) \to C \\ (a_n) \mapsto \lim_{n\to\infty} a_n \end{array} \right.$

$r < 1 \quad f_n = \sum_{k\geq 0} c_k^{(n)} z^k \in \mathcal{O}(|z|\leq r)$ bounded seq.

$\forall k \quad (c_k^{(n)})_n \in \ell^\infty(C) \quad \Rightarrow \quad \exists\ c_k = \lim_{n\to\infty} c_k^{(n)}$

$\longrightarrow \sum_{k\geq 0} c_k z^k \in \mathcal{O}(|z|\leq r') \quad \forall r' < r.$

Now start with $f_n \in \mathcal{O}(D_n)^\times$. WLOG $f_n(0) = 1$

Form $\quad g_n = \lim_{k\to\infty} (f_{n+1} \cdots f_{n+k}) \in \mathcal{O}(D_n)$

Not difficult $\quad \|g_n - 1\| < 1 \Rightarrow g_n \in \mathcal{O}(D_n)^\times$;

$g_n = f_{n+1}\, g_{n+1} \qquad \dfrac{f_n g_n}{f_{n+1} g_{n+1}} = f_n.$

<u>Drinfeld case in dim 1</u>:

$\Omega = \Omega^1_K \hat{\otimes} C = \mathbb{P}^1_C \setminus \mathbb{P}^1(K). \qquad K = \mathbb{Q}_p$

<u>Thm</u> 1) (Drinfeld) $\quad H^1_{\acute{e}t}(\Omega, \mathbb{Z}_p(1)) \xrightarrow{\sim} St^{cont}(\mathbb{Z}_p)^*$

$$St^{cont}(\mathbb{Z}_p) = \underline{\mathcal{C}^0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z}_p)}_{/\text{cont}}$$

2) (CDN) $\quad 0 \to \mathcal{O}(\Omega)/C \to H^1_{proét}(\Omega, \mathbb{Q}_p(1)) \to St(\mathbb{Q}_p)^* \to 0.$

Proof (sketch) - Kummer + $Pic(\Omega) = 0$

$\Rightarrow 1)$ $\quad$ (+ $R^1 \lim \mathcal{O}(U_n)^\times = 0$

U_n "standard covering")

2^{nd} part: need to understand $\{\mathcal{O}^\times(U_n) \hat{\otimes} \mathbb{Q}_p\}$ as pro-system

Crash course on p-adic LL.

$L/\mathbb{Q}_p$ finite

G p-adic Lie group

$$\begin{tikzcd} & \mathrm{Ban}\,G \arrow[dl] \arrow[dr] & \\ \mathrm{An}\,G \arrow[rr] & & \mathrm{Sm}\,G \end{tikzcd}$$

$\mathrm{Sm}(G)$ = adm. sm. reps of G (classical)

$$\begin{tikzcd} & \Pi \arrow[dl] \arrow[dr] & \\ \Pi^{an} \arrow[rr] & & \Pi^{sm} \end{tikzcd}$$

$\mathrm{An}(G)$ = adm. (Schneider - Teitelbaum) loc an reps of G

(on L-spaces of compact type).

$\mathrm{Ban}(G)$ = adm. unitary L-Ban reps of G

$\exists G$ - inv. open lattice $\Pi^\circ \subseteq \Pi$

$(\Pi^\circ \otimes_{\mathcal{O}_L} k_L)^H$ is finite $\forall$ compact open $H \leq G$

$\Pi \in \mathrm{Ban}(G) \rightsquigarrow \Pi^{an} = \{v \in \Pi \mid \begin{array}{l} G \to \Pi \\ g \mapsto gv \end{array}$ is loc. an$\}$

$\Pi \in \mathrm{An}(G) \rightsquigarrow \Pi^{sm} = \Pi^{\mathfrak{g}=0}$

Thm. (Schneider - Teitelbaum) Π^{an} is dense in Π and $\Pi \to \Pi^{an}$ is exact.

Rmk 1) $\Pi^{sm} = 0$ most of the time and $\Pi \to \Pi^{sm}$ not exact.

2) $\Pi \to \Pi^{an}$ is not fully faithful

($G = \mathbb{Z}_p$, $\Pi = \mathcal{C}^0(\mathbb{Z}_p, L)$:

$\mathrm{End}_G \Pi$ = measures on $\mathbb{Z}_p$

$\mathrm{End}_G \Pi^{an}$ = distributions on $\mathbb{Z}_p$).

3) Π irred. $\not\Rightarrow$ Π^{an} irred.

$\widehat{G} = \{$ irred. top. in Ban $G\}/\sim$ very poorly understand.

Ex: $G/\mathbb{Q}_p$ reductive, B parabolic

$St_P^{G\,cont}$ generalized Steinberg.

Grosse-Klonne, Conrgdy $St_P^{G, cont}$ is irred

χ unitary char of a tori of B $\to$ $(\mathrm{Ind}_B^G \chi)^{cont}$ finite length.

Naive construction:

$\Pi \in An(G)$ Is there a unitary Ban. rep $\widehat{\Pi}$ with a universal map $\Pi \to \widehat{\Pi}$? If it exists, $\widehat{\Pi}$ is called the univ. unitary completion of Π (Emerton).

- Answer: not necessarily $G = \mathbb{Z}_p$ $\Pi = LA(\mathbb{Z}_p, L)$

  $\widehat{\Pi}$ does not exist ($\widehat{\Pi}^{an} \neq \Pi$ in general).
- Even if it exists, it can very well be 0.
- $(\widehat{\Pi})^{an}$ not necessarily Π: it can be 0, or it can be of finite length adm. and yet larger than Π.

Examples:

1) Π irred. unramified $G = GL_2(\mathbb{Q}_p)$

W irred. alg. rep of G

$\widehat{\Pi \otimes W} \in \widehat{G}$ (Berger, Breuil)

2) $\widehat{St} = St^{cont}$

3) (Breuil) $k > 2$ $St \otimes W$ $W = \ldots$

$B(k) = \widehat{St \otimes W}$ not admissible.

Ω = Drinfeld space in dim 1 / $\mathbb{Q}_p$

$\mathcal{O}(k) = \mathcal{O}(\Omega) \otimes L$ with $g.f(z) = |\det g|^{1-\frac{k}{2}} \frac{\det g}{(a-cz)^k} . f\left(\frac{dz-b}{a-cz}\right)$

$B(k)^* \xrightarrow{\sim} \mathcal{O}(k)^{G\text{-bd}}$

$= \{f \in \mathcal{O}(k) \mid Gf \text{ bounded}\}$.

4) Π supercuspidal rep. of G, $\widehat{\Pi}$ not adm., but has very interesting irred. quotients.

The big results:

Colmez's functor: $\Pi \in \mathrm{Ban}(G)$

$$\Pi^*_{pc} = \{ l \in \Pi^* \mid \lim_{x \to \infty} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} l = 0 \}$$

presque compact

Stable under $\begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$

$A_{\inf}^{\mathrm{Ker}\,\chi_{cycl}} \xrightarrow{\sim}$ measures on $\mathbb{Z}_p$ μ s.t. $\mu * \delta_b \xrightarrow[b \to \infty]{} 0$

$\int_{\mathbb{Z}_p} \underset{\underset{e^{2i\pi x}}{\parallel}}{[\varepsilon^x]}\, \mu \longleftarrow \mu$

Π^*_{pc} is naturally a module over the ring of such measures, so over $A_{\inf}^{\ker \chi}$.

$$V(\Pi)^*(1) = \left(W(\mathbb{C}_p^\flat) \otimes_{A_{\inf}^{\ker\chi}} \Pi^*_{pc} \right)^{\varphi = 1} \qquad \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \text{ on } \Pi^*_{pc}$$

$\in \mathrm{Rep}_L\, G_{\mathbb{Q}_p}$.

Thm - (Colmez, Paškūnas, Dospinescu) - This induces a bij
$\widehat{G}_{cusp} \overset{1-1}{\longleftrightarrow} \{$ irred. 2-dim L-reps of $G_{\mathbb{Q}_p} \}/\sim$.

not subquotient of a parabolic induction.

Rmk - 1) If $p \geq 5$ Colmez + Paškūnas -
$\forall p$: CDP.

2) Berger-Breuil th is a key input in the proof.

3) Most delicate part: compatibility with CFT
$\det V(\Pi) = \chi.\, \omega_\Pi$

Thm (Colmez - Emerton) - $\Pi \in \widehat{G}_{cusp}$ satisfies $\Pi^{sm} \neq 0$
$\Leftrightarrow V(\Pi)$ is de Rham weights 0,1

In this case $\Pi^{sm} = LL(WD(D_{pst}(V(\Pi))))$.