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Cohomology of Shimura varieties. (A. Caraiani) (1)

Goal:

1) Strategy for computing coho. of Shimura varieties:
compact unitary Sh. varieties
"generic" part of coho. (with Scholze).

2) Emphasize connection to Fargues's conjecture:
local-global compatibility.

Plan
1) The HT period morphism π_{HT}.
2) The Newton stratification.
3) The fibers of π_{HT}.
4) The generic part of coho.

§1. The HT period morphism.

Let (G,X) be a Sh. variety
datum of Hodge type: $G/\mathbb{Q}$ connected red gp $/\mathbb{Q}$
X = conj. class of hom.

$h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \longrightarrow G_{\mathbb{R}}$ satisfying certain axioms s.t.
X is a variation of polarizable Hodge structures,
in particular Griffiths transversality is satisfied.

$\exists$ closed embedding of Shimura data:

$(G,X) \hookrightarrow (\tilde{G},\tilde{X})$ where $\tilde{G} = GSp_{2g}$ some $g \geq 1$

$\tilde{X}$ = Siegel double space

$= \left\{ Z \in M_g(\mathbb{C}) \mid Z = Z^t, \ \mathrm{im}\, Z \text{ is positive or negative definite} \right\}$

Hodge cocharacter:

$\mu = \mu_h = (h \times_{\mathbb{R}} \mathbb{C})\big|_{1^{st}\ \mathbb{G}_m \text{ factor}} : \mathbb{G}_{m,\mathbb{C}} \longrightarrow G_{\mathbb{C}}$

$\{\mu\}$ conjugacy class. G. transversality $\Rightarrow$ μ is minuscule.
$E/\mathbb{Q}$ reflex field = field of def. of $\{\mu\}$.

$K \subset G(\mathbb{A}_f)$ compact open

$\rightsquigarrow X_K / E$ Sh. variety of level K

$$X_{K,\mathbb{C}} = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K .$$

Hecke algebra $\circlearrowright H^*_{ét}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$ can be computed in terms of automorphic reps of G (Matsushima, Franke)

What can we say about $H^*_{ét}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_\ell)$?

μ determines a grading on the underlying cat. of v.s. of $\mathrm{Rep}_E G$. It determines two filtrations on $\mathrm{Rep}_E G$:

$$V \in \mathrm{Rep}_E G \rightsquigarrow \mathrm{Fil}^p_{std} V_{\mathbb{C}} = \bigoplus_{p' \geq p} V_{\mathbb{C}}^{(p',q')}$$

$$\mathrm{Fil}^p V_{\mathbb{C}} = \bigoplus_{p' \leq p} V_{\mathbb{C}}^{(p',q')}$$

$$\longrightarrow P_\mu^{std} \subseteq G_{\mathbb{C}} \rightsquigarrow \mathcal{Fl}_{G,\mu}^{std} = \left(G_{\mathbb{C}} / P_\mu^{std}\right)$$

$$P_\mu \subseteq G_{\mathbb{C}} \rightsquigarrow \mathcal{Fl}_{G,\mu} = \left(G_{\mathbb{C}} / P_\mu\right)$$

two opposite parabolics defined over E

$\exists$ holo. embedding $X \hookrightarrow \mathcal{Fl}^{std}_{G,\mu,\mathbb{C}}$ (Borel embedding)

$\mathcal{Fl}_{\tilde{G},\mu} \cong$ param. tot. isotropic g-dim v.s. inside $\mathbb{Q}^{2g}$.

$\mathfrak{P} \mid p$ prime of E

$\mathcal{Fl}_{G,\mu} = \left(G / P_\mu \otimes_E E_{\mathfrak{P}}\right)^{ad}$ HT-period domain.

$\mathcal{X}_K = (X_K \otimes_E E_{\mathfrak{P}})^{ad}$

<u>Thm 1</u> (Scholze, C.-Scholze).

1) $\forall K^p \subseteq G(\mathbb{A}_f^p)$ compact open

$\exists!$ perfectoid space $\mathcal{X}_{K^p} \sim \varprojlim_{K_p} \mathcal{X}_{K^p K_p}$.

(for diamonds, $\mathcal{X}_{K^p}^\diamond = \varprojlim_{K_p} \mathcal{X}^\diamond_{K^p K_p}$).

2) There exists a $G(\mathbb{Q}_p)$-equivariant morphism:

$$\pi_{HT} : \mathcal{X}_{K^p} \longrightarrow \mathcal{F}\ell_{G,\mu}.$$

<u>Remarks</u>: 1) In Siegel case, on $\mathcal{X}_{K^p}(L, L^+)$:

($L/\mathbb{Q}_p^{cycl}$ complete ext.)

$$(A/_L , T_pA \overset{\alpha}{\underset{\uparrow}{\cong}} \mathbb{Z}_p^{2g}) \longmapsto \mathrm{Lie}\, A \subset T_pA \otimes_{\mathbb{Z}_p} L \overset{\alpha}{\cong} L^{2g}.$$

symplectic isom.

2) π_{HT} is $G(\mathbb{A}_f^p)$-equivariant for $G(\mathbb{A}_f^p)$-action on $(\mathcal{X}_{K^p})_{K^p}$ and trivial action on $\mathcal{F}\ell_{G,\mu}$.

3) $\mathcal{X}^*_{K^p} \longleftarrow$ min. compactification

$\downarrow \pi_{HT}$

$\mathcal{F}\ell_{G,\mu}$

π_{HT} is affinoid

i.e. $\exists$ affinoid cover $\{U_i\}$ of $\mathcal{F}\ell_{G,\mu}$ s.t. $\pi_{HT}^{-1}(U_i)$ is affinoid perfectoid.

4) Upshot :

$$\begin{tikzcd} & \mathcal{X}_{K^p} \arrow[dl] \arrow[dr] & \\ \mathcal{X}_K & & \mathcal{F}\ell_{G,\mu} \end{tikzcd}$$

Hochschild-Serre spectral seq. ; Leray-Serre spectral sequence.

Reduced to understanding $R\pi_{HT*}\mathbb{F}_\ell$ on $\mathcal{F}\ell_{G,\mu}$.

$R\pi_{HT!}\mathbb{Q}_\ell$

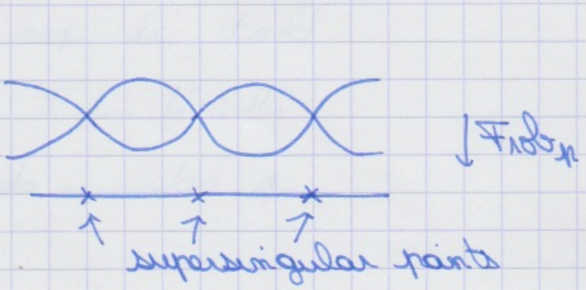
Ideas in the proof of thm 1:

1) $g = 1$ (modular curve case)

$\overline{X}_{K_0(p)} \curvearrowleft \Gamma_0(p) \cap K$

special fiber $/\mathbb{F}_p$

$\overline{X}_K$



2) In the Siegel case, π_{HT} is constructed from relative

HT-filtration for : $\pi : \mathcal{A} \longrightarrow \tilde{\mathcal{X}}_{\tilde{K}}$

$\uparrow$ universal abelian variety

"measures relative position of p-adic and dR cohomologies under comparison isom."

Consider : $\pi : \mathcal{Y} \to \mathcal{X}$ proper + smooth morphism of smooth adic spaces over $\mathrm{Spa}(L, \mathcal{O}_L)$ $L/\mathbb{Q}_p$ finite (later, take $L = E_{\mathfrak{g}}$, $\mathcal{Y} = \mathcal{A}$, $\mathcal{X} = \mathcal{X}_{\tilde{K}}$).

On $\mathcal{X}_{\text{proét}}$: $R^i\pi_{*}\mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{X}}} \widehat{\mathcal{O}}_{\mathcal{X}} \hookrightarrow R^i\pi_{HT*}\widehat{\mathcal{O}}_{\mathcal{Y}} \simeq R\pi_{*}\widehat{\mathbb{Z}}_{p,\mathcal{Y}} \otimes_{\widehat{\mathbb{Z}}_{p,\mathcal{X}}} \widehat{\mathcal{O}}_{\mathcal{X}}$

primitive comparison isom.

Consider the following sheaves on $\mathcal{X}_{\text{proét}}$:

$$\widehat{\mathcal{O}}^+_{\mathcal{X}^\flat} = \varprojlim_{\text{Frob}} (\mathcal{O}^+_{\mathcal{X}}/p)$$

$$\mathbb{B}^+_{dR,\mathcal{X}} = \text{completion of } W(\widehat{\mathcal{O}}^+_{\mathcal{X}^\flat})\left[\frac{1}{p}\right]$$

along $\ker\theta$, $\theta : W(\widehat{\mathcal{O}}^+_{\mathcal{X}^\flat})\left[\frac{1}{p}\right] \to \widehat{\mathcal{O}}_{\mathcal{X}}$

$$\mathbb{B}_{dR,\mathcal{X}} = \mathbb{B}^+_{dR,\mathcal{X}}\left[\frac{1}{\xi}\right]$$ where ξ generates $\ker\theta$ (ξ exists pro-étale locally on $\mathcal{X}$, unique up to unit).

A $\mathbb{B}^+_{dR,\mathcal{X}}$-local system on $\mathcal{X}$ is a sheaf of $\mathbb{B}^+_{dR,\mathcal{X}}$-modules that is pro-étale locally free.

$\pi : \mathcal{Y} \to \mathcal{X}$

p-adic - de Rham comparison $\Rightarrow$ two $\mathbb{B}^+_{dR,\mathcal{X}}$-local systems on $\mathcal{X}_{\text{proét}}$

$$\mathbb{M} := R^i\pi_*\widehat{\mathbb{Z}}_{p,\mathcal{Y}} \otimes_{\widehat{\mathbb{Z}}_{p,\mathcal{X}}} \mathbb{B}^+_{dR,\mathcal{X}}$$ "étale coho."

$$\mathbb{M}_0 := \left(R^i\pi_{dR*}\mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}\mathbb{B}^+_{dR}\right)^{\nabla=0}$$ "dR coho."

$$\rightsquigarrow \mathbb{M} \otimes_{B^+_{dR,x}} B_{dR,x} \simeq \mathbb{M}_0 \otimes_{B^+_{dR,x}} B_{dR,x}$$

$$\mathbb{M}_0 \subseteq \mathbb{M}.$$

The relative position of $\mathbb{M}_0$ and $\mathbb{M}$ induces a filtration on $\mathrm{Gr}^0 \mathbb{M} = R^i\pi_* \widehat{\mathbb{Z}}_{p,Y} \otimes_{\widehat{\mathbb{Z}}_{p,X}} \widehat{\mathcal{O}}_X$.

For $i=1$, get short exact sequence:

$$(*) \left(\begin{array}{l} 0 \longrightarrow R^1\pi_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X \longrightarrow R^1\pi_* \widehat{\mathbb{Z}}_{p,Y} \otimes_{\widehat{\mathbb{Z}}_{p,X}} \widehat{\mathcal{O}}_X \\ \qquad\qquad \longrightarrow \pi_* \Omega^1_Y \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X(-1) \longrightarrow 0. \end{array} \right.$$

To define π_{HT}, ~~define~~ evaluate $(*)$ on aff. perf. cover of $|\widetilde{X}_{K^p}|$.

3) For general Hodge type:

$$(G,X) \hookrightarrow (\widetilde{G}, \widetilde{X})$$

(need to show π_{HT} for (G,X) is independent of this.

$G = \mathrm{Stab}_{\{s_\alpha\}} \widetilde{G}$ $\qquad$ $\{s_\alpha\}$ finite collection of Hodge tensors.

Want: $\{s_{\alpha,p}\}$ respects HT-filtration.

$\{s_{\alpha,dR}\}$ preserves Hodge-dR filtration.

$s_{\alpha,dR}$ and $s_{\alpha,p}$ matched under p-adic-dR comparison (Blasius, Deligne).

§2. Newton stratification.

Purely local setup.

$G/\mathbb{Q}_p$ connected red group, $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \longrightarrow G_{\overline{\mathbb{Q}}_p}$ minuscule cochar.

$\{\mu\}$ conj. class $/L$, $L/\mathbb{Q}_p$ finite

$\rightsquigarrow \mathcal{F}\ell_{G,\mu}$ as before.

$\underbrace{B(G,\mu)}_{\mu\text{-admissible elements}} \subseteq B(G)$

Newton map $\nu : B(G) \longrightarrow (X_*(G) \otimes \mathbb{Q})^\Gamma$ $\quad \Gamma = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$

Kottwitz map $\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma$

(ν, κ) is injective, $B(G)$ equipped with a particular order $\leq$

μ determines $(\overline{\mu}, \mu^\flat) \in \mathrm{Im}(\nu, \kappa)$

Say $b \in B(G, \mu) \iff \begin{cases} \nu_b \geq \overline{\mu} \\ \kappa(b) = \mu^\flat \end{cases}$

<u>Thm 2</u> (C.-Scholze). There exists a Newton stratification:

$$\mathcal{F}\ell_{G,\mu} = \bigcup_{b \in B(G,\mu)} \mathcal{F}\ell_{G,\mu}^b \leftarrow \text{(locally closed strata.}$$

Moreover, $\mathcal{F}\ell_{G,\mu}^{\geq b} = \bigcup_{b' \geq b} \mathcal{F}\ell_{G,\mu}^{b'}$ is closed.

(In particular, basic locus open)

<u>Remarks</u>: 1) The stratification is highly non-algebraic

$G = GL_2$, $\mu = (1,0)$

$\mathbb{P}^{1,ad} = \mathbb{P}^1(\mathbb{Q}_p) \cup \Omega^2$

2) We don't know if this is a true stratification.

<u>Hansen</u>: for GL_n, $\overline{\mathcal{F}\ell_{G,\mu}^b} = \mathcal{F}\ell_{G,\mu}^{\geq b}$.

<u>Sketch of proof of Thm 2</u>:

~~Ingred~~ <u>Ingredients</u>:

- Fargues's classification of G-bundles on curve (Anschutz) over alg. closed perf. field.

$C/\mathbb{Q}_p$ alg. closed perf. field X_C $\quad \begin{pmatrix} F \hookrightarrow C \\ E \hookrightarrow \mathbb{Q}_p \end{pmatrix}$

$|\mathrm{Bun}_G| \cong B(G)$.

- The B_{dR}^+ Grassmannian: (Berkeley notes SW).

Define $Gr_G^{B_{dR}^+}$ to be the (pre-) sheaf on $\mathrm{Perf}_{\mathbb{F}_p}$ (with pro-étale topo).

given by functor of points:

$\mathrm{Perf}_{\overline{\mathbb{F}}_p} \ni S = \mathrm{Spa}(R, R^+) \longmapsto \mathrm{Gr}_G^{B_{dR}^+}(S)$

$$\begin{array}{l} \mathrm{Gr}_G^{B_{dR}^+}(S) \\ \downarrow \\ \mathrm{Spd}\, \breve{O}_E^{\diamond} \\ (R^\#, \mathcal{E}) \longmapsto R^\# \end{array} \quad = \left\{ \begin{array}{l} R^\# \text{ untilt of } R / \breve{O}_E \\ \mathcal{E} \quad G\text{-bundle} / \mathrm{Spec}\, B_{dR}^+(R^\#) \end{array} \right.$$

with trivialization:

$\iota : \mathcal{E}|_{\mathrm{Spec}\, B_{dR}(R^\#)} = G \times \mathrm{Spec}\, B_{dR}(R^\#)$

Can also think of $\mathrm{Gr}_G^{B_{dR}^+}$ as a sheaf / $\mathrm{Perf}_{\breve{O}_E}$.

<u>Remark</u>. Kedlaya-Liu: can glue finite proj. modules in pro-étale topology

$\Rightarrow \mathrm{Gr}_G^{B_{dR}^+}$ sheaf for pro-étale topo.

$\mu : \mathbb{G}_{m, \breve{E}} \longrightarrow G_{\breve{E}}$ determines Schubert cell

$\mathrm{Gr}_{G,\mu}^{B_{dR}^+} \subseteq \mathrm{Gr}_G^{B_{dR}^+} \otimes_{\breve{O}_E} L$.

(K, K^+) perfectoid field

$\mathrm{Gr}_G^{B_{dR}^+}(K, K^+) = G(B_{dR,K}) / G(B_{dR,K}^+)$

(uses $B_{dR,K}^+ \underset{\text{abstractly}}{\cong} K[[\xi]]$).

If $K = C$ alg. closed, Cartan decomposition:

$$G(B_{dR,C}) = \bigsqcup_{\mu' \in X_*(G)_{dom}} G(B_{dR,C}^+)(\mu')^{-1}(\xi)\, G(B_{dR,C}^+).$$

Define $\mathrm{Gr}_{G,\mu}^{B_{dR}^+}(R, R^+) = \left\{ \begin{array}{l} \mathcal{E} \text{ as before} \\ \forall\, s \underset{\text{geo. pt.}}{\longrightarrow} \mathrm{Spa}(R, R^+) \\ \mathcal{E}|_{\mathrm{Spa}(C(s), C(s)^+)} \leftrightarrow \mu \in X_*(G)_{dom} \end{array} \right.$

open Schubert cell

μ minuscule $\Rightarrow$ also closed.

<u>Prop</u>. (C.-Scholze) $\exists$ isomorphism (of ~~sheaves~~ sheaves on Perf_L / diamonds)

$$\mathcal{F}\ell_{G,\mu}^{\diamond} \xrightarrow{\sim} \mathrm{Gr}_{G,\mu}^{B_{dR}^+}.$$

Upshot :

$$\begin{tikzcd}
\mathcal{F}\ell^{\diamond}_{G,\mu} \arrow[r, "\sim"] \arrow[d] \arrow[rrr, bend left, "\mathcal{D}"] & \mathrm{Gr}^{\mathbb{B}_{dR}^+}_{G,\mu} & \subset \mathrm{Gr}^{\mathbb{B}_{dR}^+}_{G} \arrow[r] & \mathrm{Bun}_G \\
\mathrm{Bun}_G^{\kappa=\mu^\flat,\ \nu\geq\overline{\mu}} \arrow[rrru, hook] & & &
\end{tikzcd}$$

Beauville - Laszlo gluing
(Kedlaya - Liu)

$|\mathcal{D}| : |\mathcal{F}\ell_{G,\mu}| \to |\mathcal{F}\ell^{\mathrm{Berk}}_{G,\mu}| \longrightarrow B(G,\mu^{-1}) \subseteq B(G)$

Closure relations : Kedlaya - Liu for GL_n.

<u>Sketch</u> of <u>the proof</u> of <u>the proposition</u> :

1) Define $\mathrm{Gr}^{\mathbb{B}_{dR}^+}_{G,\mu} \xrightarrow{\pi_{G,\mu}} \mathcal{F}\ell^{\diamond}_{G,\mu}$

Bialynicki - Birula morphism.

reduce to $G = GL_n$.

$\mathrm{Gr}^{\mathbb{B}_{dR}^+}_{GL_n,\mu}$ parametrizes $\mathbb{B}^+_{dR,R}$ - lattices Λ inside $\mathbb{B}_{dR,R}^n$.
$(\mathrm{Spa}(R,R^+))$.

Relative position of Λ and std lattice
$\rightsquigarrow$ filtration on $R^n \cong (\mathbb{B}^+_{dR,R})^n / \xi (\mathbb{B}^+_{dR,R})^n$.

2) Show $\pi_{G,\mu}$ induces bijection on (K,K^+) - points
(K perf, $\mathbb{B}^+_{dR,K} \cong K[[\xi]]$).

3) Injectivity reduced to 2).

4) Surjectivity :
Because $\mathrm{Gr}^{\mathbb{B}_{dR}^+}_{G,\mu}$ is a sheaf, check surjectivity pro - étale locally on $\mathcal{F}\ell_{G,\mu}$.

<u>Need</u>: $V \in \mathrm{Rep}_L G \longmapsto \mathbb{M}_V \subseteq \mathbb{B}_{dR} \otimes_L V$
$\mathbb{B}^+_{dR}$ - local system which maps to right object under $\pi_{G,\mu}$.

$V \in \mathrm{Rep}_L G$ (filtered mod. with integrable connection.)

$\downarrow$

$(V \otimes_L \mathcal{O}_{\mathcal{F}\ell_{G,\mu}},\ id \otimes \nabla_{GM},\ \mathrm{Fil})$ ← universal filtration param. by $\mathcal{F}\ell_{G,\mu}$.

$\downarrow$

$\mathbb{M}_{0,V}$ dR - $\mathbb{B}_{dR}^{+}$ - local system

$\downarrow$ ← Scholze "p-adic étale local system"

$\mathbb{M}_V$ π.

Back to global setup:

$$b : \mathcal{F}\ell_{G,\mu}^{\diamond} \longrightarrow \mathrm{Bun}_G$$

$d = \dim X_K$

π = automorphic rep. of G s.t. π_p is discrete.

Local - global compatibility for Fargues's conjecture:

$$R\pi_{HT*}(\overline{\mathbb{Q}}_\ell[d])[\pi_p] = m\, b^* \underbrace{\mathcal{F}_{\varphi_{\pi_p}}}$$

(Perverse sheaf conj. by Fargues.)