06/06/2018 | § 5. Conclusion on formal moduli spaces. (1)

RZ3 | Last time: $\mathcal{D}_{\mathbb{Z}_p}$ = integral RZ-datum of EL-type or PEL-type

$(p \neq \ell)$ $\rightsquigarrow \mathcal{D}_{\mathbb{Q}_p}$ = rational – – – – – – –

$\rightsquigarrow (G, \{\mu\}, b)$ Assumption: G connected reductive group

$\begin{cases} \mathcal{G} \ \mathbb{Z}_p\text{-gp scheme} \\ E, J_b \end{cases}$

$\mathcal{D}_{\mathbb{Z}_p} \rightsquigarrow$ formal scheme $\mathcal{M}^{naive}_{\mathcal{D}_{\mathbb{Z}_p}}$ / Spf $O_{\breve{E}}$.

+ LMD

$$\begin{tikzcd} & \widetilde{\mathcal{M}}^{naive}_{\mathcal{D}_{\mathbb{Z}_p}} \arrow[dl] \arrow[dr, "\varphi"] & \\ \mathcal{M}^{naive}_{\mathcal{D}_{\mathbb{Z}_p}} & & (\mathbb{M}^{naive}_{\mathcal{D}_{\mathbb{Z}_p}})^{\wedge} \end{tikzcd}$$

Assume now that G split over tamely ramified extension of $\mathbb{Q}_p$ + assume $\mathcal{G}$ parahoric group scheme.
Then define closed subscheme $\mathbb{M}^{loc} \subset \mathbb{M}^{naive}$ (both proj. O_E-schemes)
with same generic fiber $X_{\{\mu\}}$
s.t. $\mathbb{M}^{loc}$ flat + reduced special fiber $\Rightarrow$ normal.
Here $\mathbb{M}^{loc}$ only depends on LM-triple $(G, \{\mu\}, \mathcal{G})$
Now define closed formal subscheme $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}} \subset \mathcal{M}^{naive}$
with LMD

$$\begin{tikzcd} & \widetilde{\mathcal{M}} \arrow[dl] \arrow[dr] & \\ \mathcal{M} & & \mathbb{M}^{loc} \end{tikzcd}$$

Then $\mathcal{M}$ formal $O_{\breve{E}}$-scheme, flat + normal.

$\mathcal{M}_{red}(k) = \bigcup_{w \in Adm_{\mathcal{G}}(\{\mu\})} X_w(b)$ union of ADLV $\;\; \| \;\; X_K(\{\mu\}, b)$

Here $X_w(b) \subset G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)$

defined as $\{ g \in G(\breve{\mathbb{Q}}_p)/\overbrace{\mathcal{G}(\breve{\mathbb{Z}}_p)}^{\breve{K}} \mid g^{-1} b \sigma(g) \in \breve{K} w \breve{K} \}$.

Theorem. $X_K(\{\mu\}, b) \neq \emptyset \iff [b] \in B(G, \{\mu\})$
(ind. of $\mathcal{G}$).

Thm (SW). G tame, $\mathcal{G}$ parahoric. The formal scheme $\mathfrak{M}$ only depends on $(G, \{\mu\}, b, \mathcal{G})$.

Proof relies on

Proposition. The functor $\mathfrak{X} \longmapsto \mathfrak{X}^\diamond$ from the cat. of flat + normal formal schemes / $\operatorname{Spf} O_{\breve{E}}$ to the cat. of v-sheaves on $\operatorname{Spf} O_{\breve{E}}$ is fully faithful.

$S \in \mathrm{Perf}_k$ $\quad \mathfrak{X}^\diamond(S) = \{(S^\#, \alpha : S^\# \longrightarrow \mathfrak{X}$ morph. of adic spaces $\}$.

Trivial remark: For other minuscule coweight $\{\mu\}$ of orth. group $\to$ have analogue of RZ-spaces.

§6. The RZ-tower and period morphism.

Drop the hyp. that G is tamely ramified and $\mathcal{G}$ parahoric.

Notation: $K_{\mathcal{G}} = \mathcal{G}(\mathbb{Z}_p) \subset \mathcal{G}(\mathbb{Q}_p)$ and let $\mathcal{M}_{K_{\mathcal{G}}} = \mathfrak{M}^{rig}$ ($\mathfrak{M}^{naive}$)

rigid analytic space over $\breve{E}$

Over $\mathcal{M}_{K_{\mathcal{G}}}$ have universal object (X, ι, λ).

$\rightsquigarrow$ local system $\mathcal{V}$ in $\mathbb{Q}_p$-vs on $\mathcal{M}_{K_{\mathcal{G}}}$, with its action by B + polariz. form (with values in $\mathbb{Q}_p(1)$).

Similarly local system $\mathcal{L}$ in $\mathbb{Z}_p$-vs ....

Proposition. The map $\begin{cases} \mathcal{M}_{K_{\mathcal{G}}}(\bar{\breve{E}}) \longrightarrow H^1(\mathbb{Q}_p, G) \\ \quad x \longmapsto \operatorname{diff}(\mathcal{V}_x, V) \end{cases}$

is constant, with value $d([b], \{\mu\})$. If $\mathcal{G}$ parahoric then is equal to zero and $\mathcal{L}_x \simeq \Lambda$.

Hence get 3 reductive groups: G, G', J_b

From now on: $\mathcal{G}$ parahoric. Then only (G, J_b)

* (Proposition. Let $\mathcal{G}$ parahoric. Then, if $\mathcal{M}_{K_{\mathcal{G}}} \neq \emptyset$, $[b] \in B(G, \{\mu\})$. Converse: holds if G tamely ramif.

For any open compact subgp $K \subset G(\mathbb{Q}_p)$, define rigid-analytic $\mathcal{M}_K$ as classifying space of K-level structures on $\mathcal{V}$, $\mathcal{V} \cong V \bmod K$

Proof of (*):
If $[b] \in B(G, \{\mu\})$ then $\mathcal{M}_{red}(k) \neq 0$.
By flatness, get $\mathcal{M}^{rig} \neq \emptyset$.

Equivalently: Let first $K \subseteq K_{\mathcal{G}}$. Consider finite étale covering $\mathcal{M}_K$ of $\mathcal{M}_{K_{\mathcal{G}}}$ classifying level structures $\mathcal{T} \cong \Lambda \bmod K$.
After this, take $K' \subseteq K \cap K_{\mathcal{G}}$ normal in K
then set $\mathcal{M}_K = \mathcal{M}_{K'}/(K/K')$.

Facts (i) Get tower $(\mathcal{M}_K \mid K \subset G(\mathbb{Q}_p))$ of rigid analytic spaces over $\breve{E}$, with action by Hecke correspondence of $G(\mathbb{Q}_p)$:

$$\begin{array}{ccc} & \mathcal{M}_{K \cap gKg^{-1}} & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle g} \\ \mathcal{M}_K & & \mathcal{M}_K. \end{array}$$

Also action of $J_b(\mathbb{Q}_p)$ on each individual member $\mathcal{M}_K$, both action agree on $Z_G(\mathbb{Q}_p)$.

(ii) Independent of parahoric $\mathcal{G}$ ie $\mathcal{M}_{K_{\mathcal{G}'}} = (\mathcal{M}_{\mathcal{G}'})^{rig}$.

(iii) The adic space $\mathcal{M}_K$ is smooth + partially proper over $\mathrm{Spa}\, \breve{E}$. The transition morph. are finite étale.

Change notation: Call $X_{\{\mu\}}$ on $\mathcal{F} = \mathcal{F}(G, \{\mu\})$
Let $\breve{\mathcal{F}} = \mathcal{F} \otimes_E \breve{E}$ considered as rigid-analytic space over $\breve{E}$.

(= Grassmannian defined over E smooth + G_E-homogeneous)

Assoc. to universal object $X/\mathcal{M}_K$, have Dieudonné isocrystal $\mathcal{D}$ = loc. free $\mathcal{O}_{\Pi_K}$-module + (Gauss-M. connection) + Hodge filtration:

$$0 \to \mathrm{Fil} \to \mathcal{D} \to \mathrm{Lie}\, X \to 0.$$

From framing ρ, get trivialisation

$$\rho_* : \mathcal{D} \xrightarrow{\sim} \mathbb{D}(\mathbb{X}) \otimes_{W(k)} \mathcal{O}_{\Pi_K} = V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\Pi_K}.$$

Get morphism $\breve{\pi}_K : \Pi_K \longrightarrow \breve{\mathcal{F}}$

<u>Facts</u>. (i) Extends to morph. of whole tower (Π_K) (compatible with $G(\mathbb{Q}_p)$-action by Hecke corr. on tower, trivial action on $\breve{\mathcal{F}}$).

Each $\breve{\pi}_K$ is equiv. for action of $J_b(\mathbb{Q}_p)$, where $J_b(\mathbb{Q}_p)$ acts on $\breve{\mathcal{F}}$ via $J_b(\mathbb{Q}_p) \subseteq J_b(\breve{\mathbb{Q}}_p)$
$\subseteq G(\breve{\mathbb{Q}}_p) \subseteq G(\breve{E})$.

(ii) The morphisms $\breve{\pi}_K$ are étale + partially proper, the fiber through $x \in \Pi_K(\breve{E})$ identified with $G(\mathbb{Q}_p)/K$.

(iii) The morphisms $\breve{\pi}_K$ factor through the period domain $\breve{\mathcal{F}}(G, \{\mu\}, b)^{wa} \subset \breve{\mathcal{F}}$

<u>Explanations</u>: a) period domain = adm. open subset of $\breve{\mathcal{F}}$, defined by Fontaine's weak admissibility. Condition: for Lie G get ~~...~~ $\mathrm{Ad}\, b \in \mathrm{End}(\mathrm{Lie}\, G)$

$$x \rightsquigarrow \mathrm{Ad}(\mathcal{F}_x).$$

b) Consider the Drinfeld case, rel. to $\mathbb{Q}_p$ ($B_{1/n}$ over $\mathbb{Q}_p$, $\{\mu\} = (1, 0, \dots 0)$, b).

Then $\mathcal{M}^\circ \cong \hat{\Omega}^{*n}_{\mathbb{Q}_p} \hat{\otimes}_{\mathbb{Z}_p} W(k)$.

Period morphism: $\underbrace{(\hat{\Omega}^{*n}_{\mathbb{Q}_p} \otimes W(k))^{\mathrm{rig}}}_{= \Omega^{*n}_{\breve{\mathbb{Q}}_p} = \mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p} \setminus \bigcup_{H/\mathbb{Q}_p} H} \xrightarrow{\text{inclusion}} \mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p}$

c) Consider LT-spaces, rel. to Dr
$(GL_n, \{\mu\}=(1,0,\dots,0), b)$. Then $\mathcal{X}^\circ \cong \mathrm{Spf}\, W(\bar{k})[[T_1,\dots,T_{n-1}]]$.
$(\mathcal{X}^\circ)^{rig}$ = open unit disc of dim $n-1 \longrightarrow \breve{\mathbb{P}}^{n-1}$.

Gross/Hopkins: LT-morphism surjective

Definition: Let $\breve{\mathcal{F}}^a$ = image of $\breve{\pi}_K$ (as adic space)
Then $\mathcal{F}^a \subset \mathcal{F}^{wa}$. (ind. of K).
By Colmez-Fontaine, $\mathcal{F}^a$ and $\mathcal{F}^{wa}$ have the same set of classical points.

Characterization of points of $\mathcal{F}^a$:
1) Faltings (Robba rings)
2) Hartl (Berger)
3) Fargues/Fontaine.
4) Scholze/Weinstein.

When is $\breve{\mathcal{F}}^a = \breve{\mathcal{F}}^{wa}$?
Obvious ex: $(GL_n, \{\mu\}=(1,0,\dots,0), \cancel{b})$. Assume b basic.
π surjective onto $\mathbb{P}^{n-1}$ so $\breve{\mathcal{F}}^a = \breve{\mathcal{F}}^{wa} = \breve{\mathcal{F}}$.
$(GL_4, \{\mu\}=(1,1,0,0))$.

Theorem (Chen, Fargues, Shen). Equality $\breve{\mathcal{F}}^a = \breve{\mathcal{F}}^{wa}$ holds if and only if $(G,\{\mu\},b)$ is fully HN-decomposable.
Roughly: For any $[b'] \in B(G,\{\mu\})$ not basic, the Hodge polygon and the Newton polygon touch in the interior.

Theorem. $(G,\{\mu\},b)$ is fully HN-decomposable if and only if $\{\mu\}$ is minute.
(if G quasi-split, then $\langle \mu, \sum_{i\in\mathcal{O}} w_i\rangle \leq 1$ $\forall \sigma$-orbits of fund. weights).

if and only if $\mathcal{X}_{b'}$ has dim 0, $\forall [b'] \in B(G,\{\mu\})$,
not basic.
if and only if "$\mathcal{X}_{red}$ has elementary description in terms of classical DL-varieties".

<u>Definition</u>. A local Shimura datum is a triple $(G, \{\mu\}, b)$ where G reductive group / $\mathbb{Q}_p$
$\{\mu\}$ conj. class of minuscule cochar.
$b \in G(\breve{\mathbb{Q}}_p)$.

<u>Thm</u> (SW): The tower (Π_K) only depends on the LSD associated to $\mathcal{D}_{\mathbb{Q}_p}$.

<u>Proposition</u>. The functor $X \longmapsto X^\diamond$ from semi-normal rigid-analytic spaces over $\breve{E}$ to diamonds over $\mathrm{Spd}\, \breve{E}$ is fully faithful.

§7. The SW-uniqueness theorem.
It suffices to exhibit associated v-sheaf (resp. diamond) associated to $\mathcal{L}$. (resp. Π_K).
Consider functor on Perf_k:
to S associates $(S^\#, \alpha)$ where $S^\#$ untilt of S
$\alpha : \mathcal{E}_b|_{X_{FF,S} \setminus S^\#} \xrightarrow{\sim} \mathcal{E}_1|_{X_{FF,S} \setminus S^\#}$
iso. of G-bundles,
meromorphic along $S^\#$, bounded by $\{\mu\}$.
Via action on $\mathcal{E}_b$, resp. $\mathcal{E}_1$, have action of $J_b(\mathbb{Q}_p)$
resp. $G(\mathbb{Q}_p)$ on functor.

<u>Thm</u>: This functor is repr. by a diamond Π_∞ over $\breve{E}$ and $\Pi_K = \Pi_\infty / K$.