01/06/2018
R.Z.2)

Recall from last time: (1)

Rational data: B/F, V = f.d. B-mod, $G = GL_B(V)$

$\{\mu\}$ cochar. of $G_{\overline{\mathbb{Q}}_p}$, with values in $\{0,1\}$ in V

$\rightsquigarrow E = E(\{\mu\}) \subseteq \overline{\mathbb{Q}}_p$

$b \in G(\breve{\mathbb{Q}}_p)$ s.t. $[b] \in A(G, \{\mu\})$.

Integral data: $\Lambda_0 \subsetneq \Lambda_1 \subsetneq \dots \subsetneq \Lambda_{l-1} \subsetneq \Lambda_l = \pi^{-1}\Lambda_0$

periodic chain of lattices in V, O_B-stable.

After choice of framing object $(\mathbb{X}, \iota)$, get naive RZ-space $\mathcal{M}$ over $\mathrm{Spf}\, \breve{O}_E$.

$\mathcal{M}(S) = \{(X_\bullet, \rho) \mid X_\bullet = X_0 \to X_1 \to \dots \to X_{l-1} \to X_0$ chain of isogeny of p-div gp/S O_B-linear, $\deg(X_{i-1} \to X_i) = \# \Lambda_i/\Lambda_{i-1}$).

$\rho = X_\bullet \times_S \overline{S} \xrightarrow[\text{q.i.}]{} \mathbb{X} \times_{\mathrm{Spec}\, \bar{k}} \overline{S}$ $\}$

+ Kottwitz conditions on each X_i.

LMD:

$$\widetilde{\mathcal{M}} \xleftarrow{\pi} \widetilde{\mathcal{M}} \xrightarrow{\varphi} M^{\mathrm{loc}}$$

principal homo. sp. under $\mathcal{G} \otimes O_E$

$\mathcal{G} = \underline{\mathrm{Aut}}(\Lambda_\bullet)$

$\widetilde{\mathcal{M}} = \{(X_\bullet, \rho), \alpha : H_{dR}(X_\bullet) \xrightarrow{\sim} \Lambda_\bullet \otimes O_S\}$

$\varphi(X_\bullet, \rho, \alpha) : \Lambda_\bullet \otimes O_S \xrightarrow{\varphi_\alpha} \mathrm{Lie}\, X_\bullet$

§3. Naive RZ spaces of PEL-type. $(p \neq 2)$

Rational data: A simple RZ-datum of PEL-type:

- F, B, V as before
- $(\ ,\ )$ alt forms on V, with values in $\mathbb{Q}_p$.
- Involution $*$ on B s.t. $(bv, w) = (v, b^* w)$
- $\{\mu\}$ conj. class of $G_{\overline{\mathbb{Q}}_p}$ where G = group of $\mathbb{Q}_p$-

  $G(R) = \{ g \in GL_B(V \otimes R) \mid (gv, gw) = c(g) \cdot (v, w),\ c(g) \in R^\times \}$.
- b with $[b] \in A(G, \{\mu\})$.

Conditions: $\mathbb{G}_m \xrightarrow{\mu} G_{\overline{\mathbb{Q}}_p} \xrightarrow{c} \mathbb{G}_{m, \mathbb{Q}_p}$

comp = id.

Integral data: a lattice Λ in V, $\mathcal{O}_B$-stable s.t.

$$\pi \Lambda \subseteq \Lambda^\vee \subseteq \Lambda \qquad \text{(vertex lattice)}$$

$\rightsquigarrow \mathcal{G}/\mathbb{Z}_p$: $\mathcal{G}(\mathbb{Z}_p) = \{ g \in G(\mathbb{Q}_p) \mid g\Lambda = \Lambda \}$.

This time: want to look at pairs (X, λ)

where X is as in EL case.

$\lambda : X \to X^\vee$ polarization (= isog. into the dual p-div. gp s.t. $\lambda^\vee = -\lambda$)

s.t. $\ker \lambda \subseteq X[\iota(\pi)]$, $\deg \lambda = \#(\Lambda/\Lambda^\vee)$.

Fix framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})/k$ s.t.

rational Dieudonné-module, with action of B

+ polar. form is isom. to $(V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p, b\sigma, \iota, (\,,\,))$.

Functor on Nilp$_{\breve{\mathbb{Z}}}$: $S \mapsto \{$iso. classes of $(X, \iota, \lambda, \rho) \mid$

(X, ι, λ) as before, $\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\operatorname{Spec} k} \bar{S}$

$\mathcal{O}_B$-linear s.t. $\rho^*(\lambda_{\mathbb{X}}) = c \cdot \lambda \times_S \bar{S}$ for some $c \in \mathbb{Q}_p^\times$ loc. on $S\}$.

It is repr. by formal scheme loc. f.f.t / Spf $\breve{\mathcal{O}}$.

Remarks: (i) G not automatically connected LAG

(this occurs when G related to orthog. groups: exclude these cases).

(ii) Even when G connected, $\mathcal{G}$ not always parahoric.

Ex) $B = F$, $* \neq \mathrm{Id}$, let $F_0 = F^{(*)}$.

If F/F_0 unramified then $\mathcal{G}$ parahoric.

If F/F_0 ramified, dim V even, then $\mathcal{G}$ is never parahoric unless $\Lambda^\vee = \pi\Lambda$.

Variants: a) Can replace $\mathbb{Q}_p$ by finite ext.

b) Instead of one Λ, can take "self dual periodic lattice chains".

Examples:

a) $B = F = \mathbb{Q}_p$, $* =$ trivial. Then $G = GSp(V, (\ ,\ ))$, only one $\{\mu\}$. Let b basic (only slopes $1/2$). $\Lambda = \Lambda^\vee$. Then $\mathcal{G}$ hyperspecial.

$\mathcal{M}^0_{red} = \begin{cases} \{pt\} \text{ if } \dim V = 2. \\ \text{union of } \mathbb{P}^1\text{'s if } \dim V = 4 \\ \text{incomprehensible if } \dim V \geq 6 \text{ (Oort, Li, Richartz).} \end{cases}$

b) $F = B$, $*$ non trivial, $F_0 = \mathbb{Q}_p$. Then $G = GU(V)$, $\{\mu\} = (1, 0 \ldots 0)$, b basic, $\Lambda = \Lambda^\vee$.
Then $\mathcal{M}^{naive}$ is $\breve{O}$-flat if F/F_0 unramified, not flat if F/F_0 ramified.
Replace $\mathcal{M}^{naive}$ by $\mathcal{M}^{Kra}$ if F/F_0 ramified.

If F/F_0 unramified, then $\mathcal{M}_{red}$ = union of compactified

(Vollaard, Wedhorn) $\leftarrow$ (DL-varieties for unitary grps, assoc. to Coxeter elements.

If F/F_0 ramified, then " " " " ,

(R. / _ / Wilson) $\leftarrow$ (" " " " " symplectic grps " " " " " " " .

c) $B = F$, $* \neq Id$, $F_0 = \mathbb{Q}_p$. Assume F/F_0 ramified. Let $\dim V = 2$.

. Assume V split. Then $G = GU(V)$ quasi-split gp. $\{\mu\} = (1, 0)$, $\Lambda^\vee = \pi\Lambda$. Finally, assume b basic.

Fact: Up to σ-conj., $\exists\ b_1$ s.t. $[b_1] \in B(G, \{\mu\})$ and b_2 s.t. $[b_2] \in A(G, \{\mu\}) \setminus B(G, \{\mu\})$

$\mathcal{M}_1^\circ \simeq \mathrm{Spf}\, W[\![T]\!]$, $\mathcal{M}_2^\circ = \mathrm{Spf}\, \mathbb{Z}_p$.

d) As in c), except assume now V anisotropic, $\Lambda^\vee = \Lambda$, b basic ($\Rightarrow [b] \in B(G, \{\mu\})$).

Then $\mathcal{M}^\circ \simeq \widehat{\Omega}^{\ell}_{\mathbb{Q}_p} \widehat{\otimes}_{\mathbb{Z}_p} \breve{\mathcal{O}}$

(alternative version of Drinfeld space).

§4. Local models.

Def: $F/\mathbb{Q}_p$ finite extension.

A local model triple over F $\underset{\text{def}}{=}$ $(G, \{\mu\}, \mathcal{G})$ where

- G reductive gp $/F$
- $\{\mu\}$ conj. class of cochar, minuscule ($\forall$ roots α $\langle \mu, \alpha\rangle \in \{0, \pm 1\}$).
- $\mathcal{G}$ parahoric gp scheme over $\mathcal{O}_F$.
  (in particular, $\mathcal{G}(\mathcal{O}_F) \subset G(F)$ parahoric subgp).

$\rightsquigarrow E = E(\{\mu\}) \subseteq \overline{F}$.

Assume G splits over tamely ramif. extension (for general case, see SW + L).

Can associate to LM triple a projective variety A over $k = \overline{k_E}$, with action of $\mathcal{G} \otimes_{\mathcal{O}_F} k$.

Namely, A closed subscheme of a loop group flag variety $LG'/L^+\mathcal{G}'$ associated to LAG G' over $k((t))$ and parahoric group scheme $\mathcal{G}'/k[\![t]\!]$.

Example. If $G = G_0 \otimes_{\mathbb{Z}_p} F$, G_0 Chevalley form

then $G' = G_0 \otimes_{\mathbb{Z}_p} k((t))$

A is a union of affine Schubert varieties

$$A = \bigcup_{w \in \mathrm{Adm}\{\mu\}} S_w$$

Def : A local model attached to the LM-triple $(G, \{\mu\}, \mathcal{G})$
$\overset{def}{=}$ proj. flat O_E-scheme M^{loc} with reduced special fiber,
with an action of gp scheme $\mathcal{G} \otimes_{O_F} O_E$ s.t.

- the generic fiber of $M^{loc} = X_{\{\mu\}} = G_E / P_{\{\mu\}}$ G_E-equiv.
- geometric special fiber of $M^{loc} = \mathcal{A}$ $\mathcal{G} \otimes k$-equiv.

Conjecture: A LM exists and is unique.

Facts: a) Existence known

b) In EL/PEL-cases, have closed emb.

$M^{loc}(G, \{\mu\}, \mathcal{G}) \subseteq M^{naive}$ iso in generic fiber.

$\Rightarrow$ M^{loc} normal scheme, all irred. comp. of $M^{loc} \otimes k$ are normal + CM.

Thm (HPR). Assume G_{ad} is F-simple and that in the product decomp. $G_{ad} \otimes_F \breve{F} \cong \prod \breve{G}_{ad,i}$, each factor is absolutely simple. Then M^{loc} is smooth over O_E iff either $\mathcal{G}$ is hyperspecial or $(G, \{\mu\}, \mathcal{G})$ is of unitary, resp. orthog., exotic good reduction type.

Explanation:

Unitary case: E/F ramif. quadr. ext.

$V = E/F$ hermit. vs, $G = GU(V)$,

$\mathcal{G} = \mathrm{Stab}(\Lambda)$ where $\Lambda^\vee = \begin{cases} \pi\Lambda & \dim V \text{ even} \\ \neq \pi V & \dim V \text{ odd} \end{cases}$

$\{\mu\} = (1, 0, \dots, 0)$ M^{loc} smooth.

Thm. Let G absolutely simple. Then M^{loc} has semi-stable, but not good reduction iff (upto unramified ext. F'/F) one of the following cases:

I] Drinfeld case ($G = PGL_n$, $\{\mu\} = (1, 0, \dots, 0)$, $\mathcal{G}$ parahoric)

2] Görtz case ($G = \mathrm{R}GL_n$, $\{\mu\} = (1^{(r)}, 0^{(n-r)})$, $\mathcal{G} = \mathrm{Stab}(\Lambda_0, \Lambda_1)$).

3] $G = \mathrm{R}Sp_{2n}$, $\{\mu\}$, $\mathcal{G} = \mathrm{Stab}(\Lambda_0, \Lambda_1)$

4] Faltings case $G = \mathrm{R}O_{2n}$, $\{\mu\} = \mu_{quadr}$, $\mathcal{G} = \mathrm{Stab}(\pi\text{-mod lattice})$