29/05/2018 Formal moduli spaces of p-div. groups. ①

[RZ], [RV], [SW Berkeley].

§1. The basic representation thm.

Def. Let X, Y p-div. groups / S.

a) An isogeny $f: X \to Y$ is a morphism, as fppf-~~schemes~~ sheaves s.t. ker f repr. by finite flat gp-scheme.

b) A quasi-isogeny is a global section f of $\mathbb{Z}$-sheaf $\mathcal{H}om(X,Y) \otimes \mathbb{Q}$ s.t. loc. on S $\exists$ n s.t. nf isogeny.

Lemma (Drinfeld). $\bar{S} \subset S$ nilp. thickening.
The can. map $\quad QIsog_S(X,Y) \to QIsog_{\bar{S}}(X_{\bar{S}}, Y_{\bar{S}})$
bijective.

Let $\mathcal{O}$ DVR, π uniformizer. Let
$Nilp_{\mathcal{O}} = \{ S \in (Sch/\mathcal{O}) \mid$ loc on S $\pi \mathcal{O}_S$ nilp. ideal sheaf $\}$.

If $char(\mathcal{O}) = (0,p)$ finite over $\mathbb{Z}_p$, then equiv:
$p\mathcal{O}_S$ loc. nilpotent.

Any formal scheme $\mathfrak{X} / Spf\, \mathcal{O}$ defines functor:
$Nilp_{\mathcal{O}} \longrightarrow Sets$

Here $\mathfrak{X}$ def. uniquely by this functor ("representable").

Thm. Let $k = \bar{k}$, char $k = p$. Let $W(k)$ = Witt vectors.
Fix p-div gp $\mathbb{X} / k$.
Consider the functor $\mathcal{M} : Nilp_W \to Sets$
$S \mapsto \{$iso classes of $(X, \rho)\}$

X = p-div. group over S

$\rho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{Spec\, k} \bar{S}$ quasi-isog.

where $\bar{S} = V(p\mathcal{O}_S)$.

$B(G,\{\mu\}) \subseteq A(G,\{\mu\}) \subseteq B(G) = G(\breve{\mathbb{Q}}_p)/_{b \sim b'}$ $b' = g\, b\, \sigma(g)^{-1}$.

This functor is repr. by formal scheme,
loc. formally of f.t. is
loc. at all $\bar{\mathbb{F}}_p$-valued point, $\mathfrak{X} = \mathrm{Spf}\, W[\![\underline{X}]\!]\langle\langle \underline{Y} \rangle\rangle$ / ideal

$f : \mathfrak{X} \to \mathfrak{Y}$ morph. of formal schemes, loc. noeth.

f loc. formally of f.t. $= \big($ $f_{red} : \mathfrak{X}_{red} \to \mathfrak{Y}_{red}$ loc. f.t.

§2. RZ-spaces of EL-type.

EL-Data: $\mathcal{D} = (F, B, V, \{\mu\}, [b])$ where

F = finite ext. of $\mathbb{Q}_p$

B = central div alg. $/F$

V = finite dim. B-module

$\{\mu\}$ = conj. class of cochar $\mu : \mathbb{G}_{m,\bar{\mathbb{Q}}_p} \to G_{\bar{\mathbb{Q}}_p}$

$[b] \in A(G, \{\mu\})$ $\quad$ $G = GL_B(V)$ viewed as LAG$/\mathbb{Q}_p$

$A(G,\{\mu\})$ Newton polygon of $[b]$ lies below Hodge polygon of μ.

Condition: For $\mu \in \{\mu\}$, $V \otimes \bar{\mathbb{Q}}_p = V_0 \oplus V_1$.

Integral EL-data: $\mathcal{D}_{\mathbb{Z}} = \mathcal{D}$ + maximal order $\mathcal{O}_B$ of B + $\mathcal{O}_F$-lattice $\Lambda \subseteq V$, stable under $\mathcal{O}_B$.

$\mathcal{D} \rightsquigarrow$ reflex field $E = E_{\{\mu\}} \underset{\text{def}}{=}$ field of def. of any class $\{\mu\}$.

$b \in [b] \rightsquigarrow J_b / \mathbb{Q}_p$ $\quad$ $J_b(R) = \{ g \in G(\breve{\mathbb{Q}}_p \otimes R) \mid g^{-1} b \sigma(g) = b \}$

$\breve{\mathcal{O}} = \mathcal{O}_{\breve{E}}$ - Will consider pairs (X, ι) over $S \in \mathrm{Nilp}_{\breve{\mathcal{O}}}$
where X p-div. group$/S$, $\iota : \mathcal{O}_B \to \mathrm{End}(X)$.

Impose Kottwitz condition rel. to $\{\mu\}$: $\quad$ $\mathcal{O}_E[T]$

$\mathcal{O}_S[T] \ni \mathrm{Char}(\iota(b) / \mathrm{Lie}\, X) = \mathrm{Char}(b \mid V_0)$ $\forall b \in \mathcal{O}_B$

Fix $(\mathbb{X}, \iota_{\mathbb{X}})$ as before over $k = \overline{\mathbb{F}}_p$.

Demand that rational Dieudonné-module of $\mathbb{X}$ isom. to $(V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p, b\sigma)$ with B-action.

Functor: $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}}} : S \mapsto \{$ iso-classes of (X, ι) as before + quasi-isog. $\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, k} \bar{S}$ $\}$

O_B-linear.

Really should be called $\mathcal{M}^{naive}_{\mathcal{D}_{\mathbb{Z}}}$.

Thm. The functor $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}}}$ is repres. by formal scheme, loc. formally of f. type / $\breve{O}$. Furthermore, all irred. compo. of $\mathcal{M}_{red}$ are proj. varieties / k.

The abstract group $J_b(\mathbb{Q}_p)$ acts on $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}$ via
$$(X, \iota, \rho) \mapsto (X, \iota, g \circ \rho), \quad g \in J_b(\mathbb{Q}_p).$$

Rk: Def. of $\mathcal{M}$ independent of the choice of $b \in [b]$, also of $\mathbb{X}$ (inside O_B-iso. class).

Rk: Why naive? In general, $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}$ not flat / $\breve{O}$. To remedy this, try to impose additional conditions on (X, ι) to define $\mathcal{M} \subseteq \mathcal{M}^{naive}$ flat.

Variants: a) Can replace div. alg. B by any simple central algebra, even semi-simple alg.

Can also replace Λ by periodic lattice chain.

Let $\mathcal{G} = \underline{Aut}(\Lambda) = GL(\Lambda)$: group scheme over $\mathbb{Z}_p$, with generic fiber G ~~[illegible]~~ parahoric gp scheme.

b) Can replace $\mathbb{Q}_p$ by finite ext. F: instead of p-div. group, look at p-div groups with strict O_F-action.

$(V\otimes_{\mathbb{Q}_p}\breve{\mathbb{Q}}_p, b\sigma)$ relative isocrystal, only 1 slope

<u>Example</u> (Drinfeld):

Fix $Q/\mathbb{Q}_p$, $F = Q$, $\mathrm{inv}_F(B) = \frac{1}{n}$, $\dim_B(V) = 1$

Λ essentially unique.

Then $G = B^\times$ as LAG/Q. Let $\mu = (1, 0, \dots, 0)$

w.r.t. $B^\times \otimes_Q \overline{Q} \cong GL_{n,\overline{Q}}$.

Take $[b]$ unique basic element in $A(G, \{\mu\})$.

<u>Thm</u>. $\left(\mathcal{M}^{\text{naive}}_{\mathcal{D}_{\mathbb{Z}_p}}\right)^\circ \cong \widehat{\Omega}^n_F \widehat{\otimes}_{O_F} \breve{O}$

Here $\widehat{\Omega}^n_F$ = formal Drinfeld upper half space: formal scheme, p-adic which is regular, with semi-stable reduction.

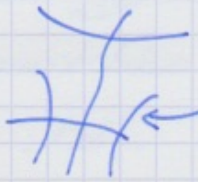
<u>Construction</u> of $\widehat{\Omega}^2_F$: Start with $\mathbb{P}^1_{O_F}$.

In $\mathbb{P}^1_{k(F)}$ $q+1$ rational points.

1st step: blow-up all those $q+1$ points, get $(\mathbb{P}^1_{O_F})^{(1)}$.

2nd step: blow-up all new $k(F)$-rat. points in special fiber of $(\mathbb{P}^1_{O_F})^{(1)}$: $(\mathbb{P}^1_{O_F})^{(2)}$.

Can continue after completion: in the limit get $\widehat{\Omega}^2_F$.



vertex in B-T tree for $PGL_2(F)$.

Almost never is $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}$ p-adic formal scheme.

<u>Example</u>. (RZ, Scholze).

Let $Q = \mathbb{Q}_p$. As before F, $\mathrm{inv}_F(B) = \frac{1}{n}$

$\dim_B(V) = 1$.

Then $G = B^\times$. Considered as $LAG/\mathbb{Q}_p$.

Fix $\varphi_0 : F \to \overline{\mathbb{Q}_p}$. Now $B^\times \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} = \prod_{\varphi : F \to \overline{\mathbb{Q}_p}} GL_{n,\overline{\mathbb{Q}_p}}$

$$\mu_\varphi = \begin{cases} (1,0,\dots,0) & \varphi = \varphi_0 \\ (1,1,\dots,1) \text{ or } (0,0,\dots,0) \end{cases}$$

Then $F \overset{\varphi_0}{\hookrightarrow} E$.

The naive RZ-space $\mathcal{M}^{naive}_{\mathcal{D}_{\mathbb{Z}_p}}$ is not flat.

RZ define closed formal subscheme $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}} \subseteq \mathcal{M}^{naive}_{\mathcal{D}_{\mathbb{Z}_p}}$ which is flat.
(defined by ideal sheaf killed by power of p).

not if $F/\mathbb{Q}_p$ unramified

Thm. (RZ, Scholze) $\mathcal{M}^{\circ}_{\mathcal{D}_{\mathbb{Z}_p}} \cong \hat{\Omega}_F \hat{\otimes}_{O_F, \varphi_0} \breve{O}$

Not explained: there exists a Weil descent datum on $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}$ from $O_{\breve{E}}$ down to O_E.

Example: $Q = F$, let $B = F$.
$\dim_F V = n$. Then $G = GL_n$ as LAG$/F$.
Let $\mu = (1, 0, \dots, 0)$.
b unique basic element of $A(G, \{\mu\})$.
$E = F$

Thm. (Lubin-Tate) $\mathcal{M}^{\circ} \cong \mathrm{Spf}\, \breve{O}[[t_1, \dots, t_{n-1}]]$
In particular $\mathcal{M}^{\circ}_{red} = \{pt\}$.

Example: (RZ) $Q = \mathbb{Q}_p$, $F/\mathbb{Q}_p$, $B = F$, $\dim V = n$
Fix $\varphi_0 : F \to \overline{\mathbb{Q}_p}$.
$G = \mathrm{Res}_{F/\mathbb{Q}_p}(GL_n)$. Take $\mu = \{\mu_\varphi\}_{\varphi : F \to \overline{\mathbb{Q}_p}}$
where μ_φ as before.

Again, $\mathcal{M}^{naive}$ not flat if $F/\mathbb{Q}_p$ ramified
$\rightsquigarrow \mathcal{M} \subseteq \mathcal{M}^{naive}$.

Thm (RZ). $\mathcal{M}^{\circ} \cong \mathrm{Spf}\, \breve{O}[[t_1, \dots, t_{n-1}]]$.

"Close" RZ-spaces are essentially the only ones we know explicitly.

Two ways of describing these formal schemes:
1) Describe explicitly $\mathcal{M}_{red}$.
$\rightsquigarrow$ theory of affine Deligne-Lusztig varieties.
2) Describe the local structure of $\mathcal{M}$ $\rightsquigarrow$ theory of local models.

Definition: Let $\mathcal{D}_{\mathbb{Z}_p}$. The associated local model proj. $\mathcal{O}_E$-scheme $\mathcal{M}^{naive}$, with point set in $Set(Sch/\mathcal{O}_E)$ given by map:

$$\varphi_\Lambda : \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S \longrightarrow t_\Lambda$$

where t_Λ is an $\mathcal{O}_B \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-module, loc. free as $\mathcal{O}_S$-mod s.t. φ_Λ is surj. + $\mathcal{O}_B \otimes \mathcal{O}_S$-linear.

Require that $\mathrm{char}(b \mid t_\Lambda) = \mathrm{char}(b \mid V_0)$ $\forall b \in \mathcal{O}_B$.

Thm There exists a LM-diagram, i.e. a diagram

$$\begin{tikzcd} & \widetilde{\mathcal{M}}^{naive} \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ \mathcal{M}^{naive} & & (\Pi^{naive})^\wedge \end{tikzcd}$$

where π torsor under $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$

φ formally smooth of relative dimension $\dim \mathcal{G}$.

In particular, $\forall x \in \mathcal{M}^{naive}(k)$ $\exists y \in$ ~~$\mathcal{M}^{naive}(k)$~~ $\Pi^{naive}(k)$ + isom. $\widehat{\mathcal{O}_{\mathcal{M},x}} \cong \widehat{\mathcal{O}_{\Pi,y}}$.

To conclude:

. the only "case" where $\mathcal{M}$ is smooth is Γ-case.

. the only case where $\mathcal{M}$ is semi-stable is Drinfeld-case.