08/06/2018 § . The Hartl–Fink curve. (1)

$K = \mathbb{F}_p((t))$, $C/\mathbb{F}_p$ alg. closed non arch. field.

$Y = \mathrm{Spa}\, K \times_{\mathbb{F}_p} \mathrm{Spa}\, C = D^*_C = \{0 < |t| < 1\}$ $\circlearrowleft \phi$

$\phi = \phi_C$

$B = H^0(Y, \mathcal{O}_Y) = \{ \sum_{n\in\mathbb{Z}} a_n t^n \mid a_n \in C, \text{ convergent for } 0<|t|<1 \}$

$\mathrm{Vect}_Y \leftrightarrow$ proj. B-modules finite rank

$\mathrm{Vect}_X \leftrightarrow$ " " " with ϕ-action

$[X = Y/\phi]$

For a reduced fraction $\frac{r}{s}$, $s > 0$,

let $\mathcal{O}(\frac{r}{s}) \leftrightarrow B^s$ with $\phi = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ t^r & & & 0 \end{bmatrix}^{-1}$

Ex: $H^0(X, \mathcal{O}_X) = B^{\phi=1} = K$

$H^0(X, \mathcal{O}_X(1)) = B^{\phi=t} = \{ \sum_{n\in\mathbb{Z}} \frac{z^{p^n}}{t^n} \mid z\in\mathcal{O}_C,\ |z|<1 \}$

Thm. Every vector bundle is $\cong$ to a $\oplus$ of $\mathcal{O}(\frac{r}{s})$.

Cor. (Fargues–Fontaine) X is "geometrically" simply connected*.

Ie every finite étale cover $Z \to X = \mathrm{Spa}\, K \times_{\mathbb{F}_p} \mathrm{Spa}\, C / \phi$ is $(\mathrm{Spa}\, A \times_{\mathbb{F}_p} \mathrm{Spa}\, C)/\phi$ where A/K is finite étale.

ie $\pi_1(\mathrm{Spa}\, K \times \mathrm{Spa}\, C/\phi) \cong \pi_1(\mathrm{Spa}\, K) \times \pi_1(\mathrm{Spa}\, C)$

$\cong \mathrm{Gal}(\overline{K}/K)$.

Proof of the corollary: If S/R is finite étale of deg d

$S \otimes_R S \to R$ is perfect

$\Rightarrow (\Lambda^d_R S)^{\otimes 2} = R$.

Let $f: Z \to X$ be finite étale of deg d,

$\mathcal{F} = f_* \mathcal{O}_Z$ vector bundle of deg d.

$\mathcal{F} \cong \oplus \mathcal{O}(\lambda_i)$ $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

After replacing K with an unramified ext.,

can assume all $\lambda_i \in \mathbb{Z}$.

$\mu : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ restricts to

$\mu_{1,1,k} : \mathcal{O}(\lambda_1) \otimes \mathcal{O}(\lambda_1) \to \mathcal{O}(\lambda_k) \rightsquigarrow H^0(X, \mathcal{O}(\lambda_k - \ell\lambda_1))$

If $\lambda_1 > 0$, $\lambda_k - \ell\lambda_1 < 0 \Rightarrow H^0 = 0$ contradicts connectedness of Z.

But $(\Lambda_X^d \mathcal{F})^{\otimes \ell} \simeq \mathcal{O}_X \Rightarrow \ell \sum \lambda_i = 0 \Rightarrow$

$\lambda_i = 0 \ \forall i$ and $\mathcal{F} \simeq \mathcal{O}_X^{\oplus d}$.

Let $A = H^0(X, \mathcal{F})$ is étale $/K$ rk d and $Z = X \times_K A$

□.

§. The Fargues–Fontaine curve.

Replace K with $\mathbb{Q}_p$?

Last time: Perf, the category of perf. spaces $/\mathbb{F}_p$ with pro-étale topology.

$\mathrm{Spd}\, \mathbb{Q}_p = \left(\mathrm{Spa}\, \mathbb{Q}_p^{\mathrm{cycl}, \flat}\right) / \underline{\mathbb{Z}_p}^*$

$= \mathrm{Spa}\, \mathbb{F}_p((t^{1/p^\infty})) / \underline{\mathbb{Z}_p}^*$

$C/\mathbb{F}_p$ alg. closed

$\mathrm{Spd}\, \mathbb{Q}_p \times \mathrm{Spa}\, C$

$= \left(\mathrm{Spa}\, \mathbb{F}_p((t^{1/p^\infty})) \times \mathrm{Spa}\, C\right) / \underline{\mathbb{Z}_p}^*$

$= \mathbb{D}_C^* / \underline{\mathbb{Z}_p}^*$

$= \varprojlim_{x \mapsto x^p} \mathbb{D}^* / \underline{\mathbb{Z}_p}^*$.

<u>Idea</u>: ~~Spd~~ "$\mathrm{Spa}\, \mathbb{Z}_p \times_{\mathbb{F}_p} \mathrm{Spa}\, \mathcal{O}_C$" $=: \mathrm{Spa}\, W(\mathcal{O}_C)$

"$\mathrm{Spa}\, \mathbb{Q}_p \times \mathrm{Spa}\, C$" $= \mathcal{Y} := \mathrm{Spa}\, W(\mathcal{O}_C) \setminus \{p[\varpi] = 0\}$

$\varpi \in C$, $0 < |\varpi| < 1$

<u>Thm</u> (Kedlaya). If $U \subseteq \mathcal{Y}$ is a rational subset, then $\mathcal{O}_{\mathcal{Y}}(U)$ is strongly noetherian.

$\Rightarrow \mathcal{Y}$ is an analytic adic space $/\mathbb{Q}_p$, with good theory of vector bundles.

Thm. $\mathcal{Y}^\diamond \cong \mathrm{Spd}\, \mathbb{Q}_p \times \mathrm{Spa}\, C$.

Proof: For $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$

$\mathcal{Y}^\diamond(S) = \{ S^\# \text{ untilt of } S,\ \text{plus } S^\# \to \mathcal{Y} \}$

$(\mathrm{Spd}\, \mathbb{Q}_p)(S) = \{ S^\# \text{ untilt of } S,\ S^\# \to \mathrm{Spa}\, \mathbb{Q}_p \}$

$(\mathrm{Spa}\, C)(S) = \{ S \to \mathrm{Spa}\, C \}$

$S^\# = \mathrm{Spa}(R^\#, R^{\#+}) \longrightarrow \mathrm{Spa}\, W(\mathcal{O}_C) \setminus \{ p[\varpi] = 0 \}$

$W(\mathcal{O}_C) \xrightarrow[\theta]{f} R^{\#+}$ s.t. $f_\theta(p), f_\theta([\varpi]) \in R^{\#*}$

$\mathcal{O}_C \longrightarrow R^{\#+}/p$ apply $\varprojlim_{x \mapsto x^p}$:

$\mathcal{O}_C \longrightarrow R^+$

$\rightsquigarrow \underbrace{\mathrm{Spa}(R, R^+)}_{S} \longrightarrow \mathrm{Spa}\, C$

Other direction: Given $S^\# / \mathbb{Q}_p$ and $S \to \mathrm{Spa}\, C$

$f : \mathcal{O}_C \longrightarrow R^+$, $\ \big\downarrow \#$

$\theta : W(\mathcal{O}_C) \to R^{\#,+}$ $\quad \theta([\alpha]) = f(\alpha)^\#$

$\longrightarrow S^\# \longrightarrow \mathcal{Y}$.

Def. $X = \mathcal{Y} / \phi_C$ the (adic) Fargues–Fontaine curve.

Thm (F.F.). Every vector bundle on X is $\cong$ to a direct sum of $\mathcal{O}(r/s)$.

$\left(|\mathrm{Bun}_{GL_n, X}| \cong B(GL_n) \right)$

Cor. $\pi_1(\mathrm{Spd}\, \mathbb{Q}_p \times \mathrm{Spa}\, C / \phi) \cong \mathrm{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_p)$

$Y = \mathrm{Spa}\, W(\mathcal{O}_C) \setminus \{p[\varpi] = 0\}$

$X = Y/_{\phi}$, $B = H^0(Y, \mathcal{O}_Y) \circlearrowleft \phi$

$H^0(X, \mathcal{O}_X) = B^{\phi=1} = \mathbb{Q}_p$

$H^0(X, \mathcal{O}_X(1)) = B^{\phi=p}$

$D_C = \{ z \mid |z-1| < 1 \}$ $\mathbb{Z}_p$-module under mult.

$\varprojlim_{x \mapsto x^p} D_C = \widetilde{D}_C$ a perfectoid space

Claim. $\widetilde{D}_C(C) \cong B^{\phi=p}$

Assume that $C = \mathbb{C}_p^{\flat}$ $\mathbb{C}_p/\mathbb{Q}_p$

then $\widetilde{D}_C \cong [\widetilde{D}_{\mathbb{C}_p}]^{\flat}$

$0 \to \mu_{p^\infty} \to D_{\mathbb{C}_p} \xrightarrow{\log} \mathbb{A}^1_{\mathbb{C}_p} \to 0$ exact seq. of sheaves of $\mathbb{Z}_p$-modules on $\mathrm{Perf}_{\mathbb{C}_p}$.

$0 \to \mathbb{Q}_p(1) \to \widetilde{D}_{\mathbb{C}_p} \to \mathbb{A}^1_{\mathbb{C}_p} \to 0$ " " " $\mathbb{Q}_p$-vs ...

In the claim $\widetilde{D}_C(C) \cong \widetilde{D}_{\mathbb{C}_p}(\mathbb{C}_p) \longrightarrow B^{\phi=p}$.

Given $(x_0, x_1, \dots) \in \widetilde{D}_{\mathbb{C}_p}(\mathbb{C}_p) \longrightarrow x \in C$

$$[x] \in W(\mathcal{O}_C) \rightsquigarrow \log[x] = \sum_{n \geq 1} (-1)^{n-1} \frac{([x]-1)^n}{n} \in B^{\phi=p}.$$

$$\phi(\log[x]) = \log[x]^{\phi} = \log[x]^p = p \log[x].$$

§ Banach-Colmez spaces.

Given $S \in \mathrm{Perf} \rightsquigarrow X_S = \left(\mathrm{Spa}\, W(R^\circ) \setminus \{p[\varpi] = 0\}\right)/_{\phi}$

$S = \mathrm{Spa}\, R$

Given $\frac{r}{s}$ a slope, get a sheaf on Perf:

$$BC(\mathcal{E}) := S \mapsto H^0(X_S, \underbrace{\mathcal{O}(r/s)}_{\mathcal{E}}).$$

<u>Thm</u>. $BC(\mathcal{E})$ is a diamond.

<u>Ex</u>) . $\mathcal{E} = \mathcal{O}(1) \Rightarrow BC(\mathcal{O}(1)) = \widetilde{D}_C$ is a perfectoid space.

. $\mathcal{E} = \mathcal{O}(2)$.

Consider $\begin{cases} \mathrm{Perf}_C \longrightarrow \mathbb{Q}_p\text{-vect} \\ S \longmapsto H^0(X_S, \mathcal{O}(2)) = B_S^{\phi = p^2}. \end{cases}$

Fix two sections $a, b \in H^0(X_C, \mathcal{O}(1))$ with distinct zeroes.

$$0 \to \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{O}(2) \longrightarrow 0 \quad \text{exact sequence.}$$
$$1 \mapsto (b, a) \qquad (x, y) \longmapsto ax - by$$

On H^0:

$$0 \longrightarrow \underline{\mathbb{Q}_p} \longrightarrow \widetilde{D}_C \oplus \widetilde{D}_C \longrightarrow BC(\mathcal{O}(2)) \longrightarrow 0$$
($BC(\mathcal{O}(2))$ ← diamond)

$$BC(\mathcal{O}(\tfrac{1}{2})) = H^0(X, \mathcal{O}(\tfrac{1}{2})) \cong \widetilde{G}$$
$G = p$-div gp ht 2, dim 1 connected.

$$0 \to \mathcal{O} \xrightarrow{t} \mathcal{O}(1) \longrightarrow i_{\infty *} \mathbb{C}_p \longrightarrow 0$$
$$0 \to \mathcal{O}(-1) \to \mathcal{O} \longrightarrow i_{\infty *} \mathbb{C}_p \longrightarrow 0$$
$$0 \to \mathbb{Q}_p \to \mathbb{C}_p \to H^1(X, \mathcal{O}(-1)) \to 0$$
($\infty \in X$ has residue field $\mathbb{C}_p$.)

$$BC(\mathcal{E}[1]) : S \mapsto H^1(X_S, \mathcal{E})$$

<u>Ex</u>) $BC(\mathcal{O}(-1)[1]) = \mathbb{A}^{1,\diamond} / \underline{\mathbb{Q}_p}$

<u>Def</u>: The category of BC spaces is the thick abelian subcategory of $\underline{\mathbb{Q}_p}$-vs sheaves on $\mathrm{Perf}_{\mathbb{C}_p}$ generated by $\underline{\mathbb{Q}_p}$, $\mathbb{A}^1 = \mathbb{G}_a$.

($\mathcal{F} \in \mathrm{Sh}(\mathrm{Perf})$, $\underline{\mathbb{Q}_p} \times \mathcal{F} \to \mathcal{F}$)

Thm (Le Bras). BC spaces are equivalent to the subcategory of $D(\mathrm{QCoh}_{X_C})$ consisting of $\mathcal{E}_{\geq 0} \oplus \mathcal{E}_{<0}[1]$ ($\mathcal{E}_{<0}$: vector bundle)

Application.

Thm. (Fargues, Scholze) $G/\mathbb{Q}_p$ red. group. Bun_G is a smooth Artin stack in diamonds. Has chart consisting of $Z \longrightarrow \mathrm{Bun}_G$ (Z smooth diamond, the map smooth)

Let $\mathcal{M}^{(S)} = \{0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0\}$ in X_S

Get
$$\begin{cases} \mathcal{M} \longrightarrow \mathrm{Bun}_{GL_2} \\ \{0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0\} \longmapsto \mathcal{E} \end{cases},$$
image in $|\mathrm{Bun}_{GL_2}|$ is $\{\mathcal{O} \oplus \mathcal{O}(1), \mathcal{O}(\tfrac{1}{2})\}$.

$$\mathcal{M} = \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}) = H^1(X, \mathcal{O}(-1)) = \mathbb{A}^1/\underline{\mathbb{Q}_p}.$$

$$\begin{array}{ccc} \Omega/\mathbb{Q}_p & \longrightarrow & \mathrm{Bun}_{GL_2}^{\mathcal{O}(1/2)} = [*/D^*] \\ \downarrow & & \downarrow \text{open} \\ \mathbb{A}^1/\mathbb{Q}_p & \longrightarrow & \mathrm{Bun}_{GL_2} \\ \uparrow & & \uparrow \\ \{0\} & \longrightarrow & \mathrm{Bun}_{GL_2}^{\mathcal{O} \oplus \mathcal{O}(1)} \end{array}$$

$D/\mathbb{Q}$ quaternions

$\Omega \longrightarrow \Omega/\mathbb{Q}_p \longrightarrow [*/D^*]$
corresponds to Drinfeld tower $\Omega_\infty \xrightarrow{D^*} \Omega$.