Adic spaces are locally $\mathrm{Spa}(R, R^+)$, R is Huber

($\underline{\text{Ex}}$: $(p) \subseteq \mathbb{Z}_p \subseteq \mathbb{Q}_p$ $\longleftarrow$) (topo. ring, $I \subseteq R_0 \subseteq R$, I fin. gen., R_0 open subring, I-adic topo.)

and $R^+ \subseteq R$ open + int. closed subring.

$\mathrm{Spa}\, R := \mathrm{Spa}(R, R^\circ)$ $\qquad$ $R^\circ \subseteq R$ power-bounded elements.

$\underline{\text{Ex}}$ $\quad \mathrm{Spa}\, \mathbb{F}_p = \{s\}$

$\mathrm{Spa}\, \mathbb{F}_p((t)) = \{\eta\} \qquad 0 < |t(\eta)| < 1$

$\mathrm{Spa}\, \mathbb{F}_p[[t]] = \{s, \eta\}$

R is Tate if it contains a topo. nilp. unit ϖ = pseudo-unif.

$$\mathrm{Spa}\, \overbrace{\mathbb{F}_p((t))}^{K_1} \times_{\mathrm{Spa}\,\mathbb{F}_p} \mathrm{Spa}\, \overbrace{\mathbb{F}_p((u))}^{K_2} = X \setminus \{|tu| = 0\}$$

$$\subseteq \mathrm{Spa}\, \mathbb{F}_p[[t]] \times_{\mathrm{Spa}\,\mathbb{F}_p} \mathrm{Spa}\, \mathbb{F}_p[[u]] = \underbrace{\mathrm{Spa}\, \mathbb{F}_p[[t,u]]}_{:= X}$$

$$\mathrm{Spa}\, K_1 \times_{\mathbb{F}_p} \mathrm{Spa}\, K_2 = \mathbb{D}^*_{\substack{K_1 \\ \mathrm{var}=u}} \cong \mathbb{D}^*_{\substack{K_2 \\ \mathrm{var}=t}}$$

$\mathbb{D}$ = open unit disc

$$\begin{cases} X \setminus \{|tu| = 0\} \longrightarrow (0, \infty) \\ x \longmapsto \dfrac{\log|t(x)|}{\log|u(x)|} \end{cases}$$

$X \setminus \{s\}$ is analytic $\qquad$ (where $|t(s)| = |u(s)| = 0$)

(= locally $\mathrm{Spa}(R, R^+)$, R is Tate)

$$X \setminus \{s\} = \underbrace{\{|t| \leq |u| \neq 0\}}_{= \mathrm{Spa}\, \mathbb{F}_p((u))\left\langle \frac{t}{u} \right\rangle} \cup \{|u| \leq |t| \neq 0\}$$

A perfectoid ring $/\mathbb{F}_p$ is a perfect complete Tate ring.

If $R/\mathbb{F}_p$ is perfectoid, $\varpi \in R$ pseudo-unif.

then $\mathbb{F}_p((\varpi^{1/p^\infty})) \subseteq R$.

$\rightarrow$ Perf, the category of perf. spaces / $\mathbb{F}_p$

Ex: $X = \mathrm{Spa}\, \mathbb{F}_p [\![t^{1/p^\infty}]\!] \times_{\mathrm{Spa}\, \mathbb{F}_p} \mathrm{Spa}\, \mathbb{F}_p [\![u^{1/p^\infty}]\!] \setminus \{s\}$

$X \in$ Perf, but does not lie over Spa (perf. field)

$X \setminus \{|tu| = 0\} = \widetilde{D}^*_{K_1} \simeq \widetilde{D}^*_{K_2}$. $\quad K_1 = \mathbb{F}_p((t^{1/p^\infty}))$

$K_2 = \mathbb{F}_p((u^{1/p^\infty}))$

Let S = profinite set, get sheaf on Perf.

$\underline{S}(X) = \mathrm{Hom}_{\mathrm{cont}}(|X|, S) \qquad (\mathrm{Spa}\, \mathbb{F}_p \notin \mathrm{Perf})$

$\underline{S}$ is not representable, but $\forall X \in$ Perf

$\underline{S} \times X$ is. If $X = \mathrm{Spa}\, R$, $S \times X = \mathrm{Spa}\, \mathcal{C}^0(S, R)$

The Diamond Spd $\mathbb{Q}_p$.

$$\begin{cases} \mathrm{Perf}_{\mathbb{Q}_p} \xrightarrow{\text{tilt}} \mathrm{Perf} \\ \quad X \longmapsto X^\flat \end{cases} \quad \leftarrow \text{not an equivalence.}$$

<u>Thm</u>. If $X \in \mathrm{Perf}_{\mathbb{Q}_p}$ then $\begin{cases} \mathrm{Perf}_X \xrightarrow{\sim} \mathrm{Perf}_{X^\flat} \\ \quad Y \longmapsto Y^\flat \end{cases}$

Ex) $\cdot\, K_1 = \mathbb{Q}_p^{\mathrm{cycl}} := \mathbb{Q}_p(\mu_{p^\infty})^\wedge$, $K_2 = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$

$K_1^\flat \cong K_2^\flat \cong \mathbb{F}_p((t^{1/p^\infty}))$

$u - (\zeta_p, \zeta_{p^2}, \dots) \qquad (p, p^{1/p}, \dots) \qquad t$

$\cdot\, X = \mathrm{Spa}\, \mathcal{O}_K [\![u^{1/p^\infty}]\!] \setminus \{|pu| = 0\} = \widetilde{D}^*_K$

(where $K = K_1$)

$t \leftrightarrow u \quad X^\flat = \underset{\text{var } u}{\widetilde{D}^*_{K^\flat}} \simeq \underset{\text{var } t}{\widetilde{D}^*_{\mathbb{F}_p((u^{1/p^\infty}))}}$

$\rightarrow$ Idea: introduce an object Spd $\mathbb{Q}_p$ s.t. TFAE:

- $\mathrm{Perf}_{\mathbb{Q}_p}$
- $X \in$ Perf with $X \longrightarrow \mathrm{Spd}\, \mathbb{Q}_p$

Given $X \in \mathrm{Perf}$, an untilt is a $\underset{(\text{perf})}{X^\#}$ with $X^{\#\flat} \simeq X$. (2)

Given $X = \mathrm{Spa}\, R$, $X^\# = \mathrm{Spa}\, R^\#$, $R^\# / \mathbb{Q}_p$.

Hint: $\mathbb{Q}_p$ is not perfectoid but $\mathbb{Q}_p^{cycl}$ is. $\mathbb{Z}_p^* \curvearrowright \mathbb{Q}_p^{cycl}$

$$\mathbb{Q}_p^{cycl,\flat} \simeq \mathbb{F}_p((t^{1/p^\infty})) \circlearrowleft \mathbb{Z}_p^* \ni \gamma \quad \gamma(t) = (1+t)^\gamma - 1$$

$$\text{“}\mathbb{Q}_p^\flat\text{”} = \text{“}\mathbb{F}_p((t^{1/p^\infty}))^{\mathbb{Z}_p^*}\text{”} = \mathbb{F}_p.$$

Let $\tilde{R}^\# = R^\# \hat{\otimes}\, \mathbb{Q}_p^{cycl} \circlearrowleft \mathbb{Z}_p^*$

$\tilde{X}^\# = \mathrm{Spa}\, \tilde{R}^\# \longrightarrow \mathrm{Spa}\, R^\#$ is a $\underline{\mathbb{Z}_p}^*$-torsor for pro-étale topo.

Let $\tilde{X} = \tilde{X}^{\#\flat}$ then $\tilde{X} \to X$ is a pro-étale $\underline{\mathbb{Z}_p}^*$-torsor.

$\tilde{X} = \mathrm{Spa}\, \tilde{R}$ where $\tilde{R} \ni t$ a pseudo-unif. st.

$\forall \gamma \in \mathbb{Z}_p^* \quad \gamma(t) = (1+t)^\gamma - 1 \quad (*)$

<u>Proposition</u>. Untilts of R to $\mathbb{Q}_p$

$\simeq \underline{\mathbb{Z}_p}^*$-torsors $\tilde{R}/R$, $t \in \tilde{R}$ st. $(*)$ holds

Given $\tilde{R}/R$ as in the proposition, $\mathbb{F}_p((t^{1/p^\infty})) \subseteq \tilde{R}$

$\mathbb{Z}_p^* \curvearrowright \mathbb{Q}_p^{cycl} \subseteq \tilde{R}^\#$

$R^\# := (\tilde{R}^\#)^{\mathbb{Z}_p^*}$.

(Prop. relies on $H^i(X_{proét}, \mathcal{O}_X^+) \overset{a}{=} 0 \ \forall i > 0$).
$X = \mathrm{Spa}(R, R^+)$

Precision: $\underline{\mathbb{Z}_p}^*$-torsor $\tilde{R}/R$ with $\mathbb{Q}_p^{cycl,\flat} \longrightarrow \tilde{R}$
$\underline{\mathbb{Z}_p}^*$-equivariant.

<u>Rmk</u>: In general, a Tate ring R is perfectoid if it's uniform and if it contains $\varpi \in R^\circ$ pseudo-unif, $\varpi^p | p$ st. $\Phi : R^\circ/\varpi \xrightarrow{\sim} R^\circ/\varpi^p$

$\underset{\text{bounded}}{R^\circ \subseteq R}$.

Recall that Perf carries the proétale topology.
Given $X \in \mathrm{Perf}$, get presheaf $h_X(Y) = \mathrm{Hom}(Y, X)$.
In fact, h_X is a sheaf.

$$\begin{cases} \mathrm{Perf} \hookrightarrow \mathrm{Sh}(\mathrm{Perf}) \\ \quad X \longmapsto h_X = X \end{cases}$$

For $\mathcal{F} \in \mathrm{Sh}(\mathrm{Perf})$ $\mathcal{F}(Y) = \mathrm{Mor}(Y, \mathcal{F})$.

<u>Def</u>: $\mathrm{Spd}\, \mathbb{Q}_p := \mathrm{Spa}\, \mathbb{Q}_p^{cycl, b} / \underline{\mathbb{Z}_p^*}$

<u>Rmk</u>: $\mathrm{Spd}\, \mathbb{Q}_p = \mathrm{Spa}\, \mathbb{C}_p^b / \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$

$= \mathrm{Spa}\, \mathbb{Q}_p(p^{1/p^\infty})^{\wedge, b} / R$

descent datum

<u>Thm</u>: Given $X \in \mathrm{Perf}$ - TFAE:

- $X \to \mathrm{Spd}\, \mathbb{Q}_p$
- $X^\# / \mathbb{Q}_p$

<u>Def</u>: A diamond D is an object in $\mathrm{Sh}(\mathrm{Perf})$ of the form $D = X/R$, where $X \in \mathrm{Perf}$ and $R \subseteq X \times X$ is a representable equivalence relation, with each $R \rightrightarrows X$ pro-étale.

~~Prop~~ If G = loc. profinite group $\quad \underline{G} \circlearrowright X \in \mathrm{Perf}$

$\underline{G} \times X \to X$

Consider $R = \underline{G} \times X \xrightarrow[\text{(action, proj)}]{} X \times X$.

Assume G acts freely on $X(C)$, alg. closed C
(then $R \hookrightarrow X \times X$).

Then $X/\underline{G}$ is a diamond.

C-points of $\mathrm{Spa}\, \mathbb{Q}_p^{cycl, b}$ are $\mathfrak{m}_C \setminus \{0\}$, $\mathbb{Z}_p^*$ acts freely

<u>Cor</u>: Untilts of C over $\mathbb{Q}_p$
$\cong (\mathfrak{m}_C \setminus \{0\}) / \mathbb{Z}_p^*$.

Prop. Given $D \in Sh(Perf)$, D is a diamond

$\Leftrightarrow \exists$ q-~~profinite~~ proétale surjection $X \to D$, $X \in Perf$

Quasi-~~profinite~~ proétale:

$$\begin{tikzcd} X \times_D Y \arrow[r, "\text{perf. qpé}"] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & D \end{tikzcd}$$

totally disc.

(Df: $$\begin{tikzcd} X \times_Y Y' \arrow[r, "\text{pét}"] \arrow[d] & Y' \arrow[d] \\ f : X \arrow[r] & Y \end{tikzcd}$$ str. tot. disc. qpé.)

$\Rightarrow$ S = loc. profinite

$$\begin{tikzcd} \underline{S} \times \mathrm{Spa}\, C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Spa}\, C \arrow[d] \\ X \arrow[r] & D \end{tikzcd}$$

The functor $X \mapsto X^\diamond$:

Let X = analytic adic space / $\mathbb{Z}_p$

(includes: rigid spaces / $\mathbb{Q}_p$ or in char p)

Def: Given $S \in Perf$,

$X^\diamond(S) = \{(S^\#, f),\ S^\#$ untilt of X, $f : S^\# \to X\}/\sim$

If $X = \mathrm{Spa}\, \mathbb{Q}_p$ recovers $\mathrm{Spd}\, \mathbb{Q}_p$.

$\mathrm{Spd}(R, R^+) := \mathrm{Spa}(R, R^+)^\diamond$.

Thm. $X^\diamond$ is a diamond.

Rmk. If $X \in Perf$ $\quad X^\diamond = X$

If $X \in Perf_{\mathbb{Q}_p}$ $\quad X^\diamond = X^\flat$

Lemma. X is pro-étale locally perfectoid.

Ex: $\quad X = \mathrm{Spa}\, \mathbb{Q}_p \langle T, T^{-1} \rangle$

$\mathbb{Z}_p^* \ltimes \mathbb{Z}_p(1) = G \quad \uparrow$

$\tilde{X} = \mathrm{Spa}\, \mathbb{Q}_p^{cycl} \langle T^{1/p^\infty}, T^{-1/p^\infty} \rangle$

$X^\diamond = \tilde{X}^\flat / \underline{G}$

Thm. $\left\{\begin{array}{l}\text{semi-normal} \\ \text{analytic adic} \\ \text{spaces}/\mathbb{Z}_p K\end{array}\right\} \longrightarrow$ diamonds / $\mathrm{Spd}\, K$

K = nonarch. field.

$$X \longmapsto X^{\diamond}$$

is fully faithful.