The pro-étale and V-topologies.

I - Intro. to the pro-étale topo: étale coho. of rigid analytic spaces.

a - $K(\pi,1)$ - spaces in rigid geometry.

Recall: A connected ~~space~~ scheme X with a generic point $\bar{x}$ is a $K(\pi,1)$ if for any local system $\mathbb{L}$ on X and all $i \geq 0$, one has $H^i(\pi_1(X,\bar{x}), \mathbb{L}_{\bar{x}}) \cong H^i(X_{et}, \mathbb{L})$

Artin: Any smooth scheme / $\mathbb{C}$ can be covered by $K(\pi,1)$ - schemes (Zariski covering).

What about rigid analytic spaces?

Th: Let $X = \mathrm{Sp}(A, A^+)$ be an affinoid rigid space connected + generic point $\bar{x}$.

Then for any $i \geq 0$ and any p- étale local system $\mathbb{L}$ on X, one has $H^i(\pi_1(X,\bar{x}), \mathbb{L}_{\bar{x}}) \cong H^i(X_{et}, \mathbb{L})$.

Rk: No "smallness" assumption, no smoothness assumption.

Pf: Want: $\forall i \geq 0$ $H^i(X_{fet}, \mathbb{L}) \cong H^i(X_{et}, \mathbb{L})$.

Let $f: X_{et} \to X_{fet}$. Enough to prove $f_* \mathbb{L} = \mathbb{L}$ and $R^j f_* \mathbb{L} = 0$ $\forall j > 0$.

ie $\forall U \to X$ fet, any coho. class in $H^j(U_{et}, \mathbb{L})$ is killed after taking a f.ét. covering of U.

Can assume U connected, can assume $\mathbb{L} = \mathbb{F}_p$.

Let $\tilde{U}$: universal cover of U, ie the inverse limit over all f.ét. covers of U - $\tilde{U} = \mathrm{Spa}(R, R^+)$ is affinoid perf.

(because taking p^{th}- roots in char 0 is étale).

Also $H^j(\tilde{U}_{et}, \mathbb{F}_p) = \varinjlim_i H^j(U_{i,et}, \mathbb{F}_p)$.

Need to check: $H^j(\tilde{U}_{et}, \mathbb{F}_p) = 0$.

Let $\tilde{U}^b = \mathrm{Spa}(R^b, R^{b+})$.

We have the Artin-Schreier exact sequence:

$$0 \to \mathbb{F}_p \to \mathcal{O}_{\tilde{U}^\flat} \to \mathcal{O}_{\tilde{U}^\flat} \to 0.$$

Because $\tilde{U}^\flat$ is perfectoid affinoid, $\forall k>0$

$$H^k(\tilde{U}^\flat_{ét}, \mathcal{O}_{\tilde{U}^\flat}) = 0.$$

$\Rightarrow \forall j>1 \quad H^j(\tilde{U}_{ét}, \mathbb{F}_p) = 0$

And $0 \to \mathbb{F}_p \to R^\flat \to R^\flat \to H^1(\tilde{U}_{ét}, \mathbb{F}_p) \to 0$

We know that $\tilde{U}^\flat$ has a non-trivial f. étale cover as $\tilde{U}^\flat_{fét} \cong \tilde{U}_{fét}$ (almost purity thm)

$\Rightarrow \begin{cases} R^\flat \to R^\flat \\ x \mapsto x^p - x \end{cases}$ is surjective and so $H^1(\tilde{U}_{ét}, \mathbb{F}_p) = 0$

$\square$

<u>Rk</u> Achinger proved that any such X is a $K(\pi, 1)$.

b- Finiteness of étale coh. of proper smooth rigid spaces

Cartan-Serre: Let X be a compact complex analytic sp. and $\mathcal{F}$ a coherent sheaf on X.

Want: $\forall i \geq 0 \quad H^i(X, \mathcal{F})$ is finite dim / $\mathbb{C}$.

Any Stein cover of X is $\mathcal{F}$-acyclic, can compute $H^*(X, \mathcal{F})$: it is the coh of the Čech complex attached to any Stein covering of X.

X being compact, can choose $\underbrace{(U_i)_{i\in I}}_{=:\mathcal{U}}, \underbrace{(V_i)_{i\in I}}_{=:\mathcal{V}}$ two Stein coverings of X, with I finite and $\forall i \in I \quad \overline{U_i} \subset V_i$.

Know: $\forall m \geq 0$ the restriction map

$$H^m(\mathcal{V}, \mathcal{F}) \xrightarrow{r_m} H^m(\mathcal{U}, \mathcal{F}) \text{ is an isom.}$$

ie $Z^m(\mathcal{U}, \mathcal{F}) = \mathrm{im}(Z^m(\mathcal{V}, \mathcal{F})) + \mathrm{im}\, d_{\mathcal{U}}^{m-1}$.

Let $E = Z^m(\mathcal{V}, \mathcal{F}) \oplus C^{m-1}(\mathcal{U}, \mathcal{F})$

$u = (r_m, 0): E \to Z^m(\mathcal{U}, \mathcal{F})$ * $u+v$ surjective

$v = (0, d_{m-1}): E \to Z^m(\mathcal{U}, \mathcal{F})$ * u completely cont.

$\dim H^m(\mathcal{U}, \mathcal{F}) < \infty$ $\Leftarrow$ * $w = (u+v) - u$ has image with finite codim. $\square$

~~Exe~~: Show finiteness of $H^i(X_{\bar{C}}, \mathbb{F}_p)$, (2)

for X proper smooth rigid space / C (complete alg closed ext. of $\mathbb{Q}_p$)

Prove at the same time that

$\dim_{\mathbb{F}_p} H^i(X_{\bar{C}}, \mathbb{F}_p) < \infty$ and that $H^i(X_{\bar{C}}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \cong H^i(X_{\bar{C}}, \mathcal{O}_X^+/p)$.

Pb: Don't have any like Stein coverings.

Lemma: Let $E^{p,q}_{*,(i)} \Rightarrow M^{p+q}_{(i)}$ be precedent spectral seq. for $i = 1, \dots, N$.

Assume: for some $r \geq 0$, the image in the r^{th} sheet of $E^{p,q}_{r,(i)} \longrightarrow E^{p,q}_{r,(i+1)}$ is an almost f.g. $\mathcal{O}_C$-mod. $\forall i, p, q$.

Then $\mathrm{im}(M^{\ell}_{(1)} \longrightarrow M^{\ell}_{(N)})$ is an almost fg. $\mathcal{O}_C$-mod if $k \leq N - \ell$.

Let $j \geq 0$. Choose $N = j + \ell$, aff. covering $(U_i^{(N)})_{i \in I} \dots (U_i^{(1)})_{i \in I}$ of X such that:

- I finite.
- $\forall i \in I$, $\forall k = 1, \dots, N-1$, $\overline{U_i^{(k)}} \subseteq U_i^{(k+1)}$
- $\forall i, \forall k$, $U_i^{(k)} \xrightarrow[\text{étale}]{} \mathbb{T}^n = \mathrm{Spa}(C\langle T_i^{\pm 1}\rangle, \mathcal{O}_C\langle T_i^{\pm 1}\rangle)$

The Čech spectral sequence $\forall k = 1, \dots, N = j + \ell$

$$E^{p, j-p}_{1,(k)} = \bigoplus_{\substack{|J| = p+1 \\ J \subseteq I}} H^{j-p}(U^{(k)}_{J, \bar{C}}, \mathcal{O}_X^+/p) \Rightarrow H^j(X_{\bar{C}}, \mathcal{O}_X^+/p)$$

Lemma $\Rightarrow$ to prove $H^j(X_{\bar{C}}, \mathcal{O}_X^+/p)$ is almost fg, enough to prove:

(*) Prop: Let V be a rational subset of X, V étale / $\mathbb{T}^n$ for some n. Let V' be a rational subset of V s.t. $\overline{V'} \subseteq V$. Then the image of $H^i(V_{\bar{C}}, \mathcal{O}_V^+/p) \longrightarrow H^i(V'_{\bar{C}}, \mathcal{O}^+/p)$ is almost f.g.

To prove this, introduce the pro-étale site.

Def. Let $f : Y \to X$ morphism of adic spaces $/C$

(1) f is said to be affinoid pro-étale if

$Y = \mathrm{Spa}(S, S^+)$, $X = \mathrm{Spa}(R, R^+)$

and $Y = \varprojlim_i Y_i \longrightarrow X$ cofiltered inverse limit of aff. adic spaces Y_i

s.t. $\forall i$ $Y_i \longrightarrow X$ is étale.

(2) f is pro-étale if it is locally aff. proétale on source and target.

morphism $(\nu : X_{proét} \to X_{ét})$

Ex. 1) Any étale morphism is pro-étale.

2) If $Z \subseteq X$ Zar. closed subset, that is

if $Z = \{x \in X, |f(x)| = 0 \ \forall f \in I\}$, $I \subseteq \mathcal{O}(X)$ ideal

$Z = \bigcap_{f_i \in I} U_{f_1, \dots f_n}$ $\qquad U_{f_1, \dots f_n} \longrightarrow X$ étale

$Z \longrightarrow X$ is pro-étale.

Def. Let X be an adic space $/C$.

The (small) pro-étale site of X is the Grothendieck topo. in the category of pro-étale morphisms $Y \to X$ s.t. a family of maps $\{Y_i \xrightarrow{f_i} Y, i \in I\}$ is a covering if $\forall i \in I$ f_i pro-étale and $\forall$ qc open $U \subseteq Y$ there exists $J \subseteq I$ finite, $\forall i \in J$ $U_i \subset Y_i$ qc open s.t. $U = \bigcup_{i \in J} f_i(U_i)$

Ex) . $\tilde{\mathbb{T}}^n = \mathrm{Spa}(C\langle T_i^{\pm 1/p^\infty}\rangle, \mathcal{O}_C\langle T_i^{\pm 1/p^\infty}\rangle) \longrightarrow \mathbb{T}^n$

pro-étale covering of gp $\mathbb{Z}_p^n$.

. $\mathbb{D} = \mathrm{Spa}(C\langle T\rangle, \mathcal{O}_C\langle T\rangle)$, then

$\{*\} \cup \mathbb{D} \setminus \{*\} \longrightarrow \mathbb{D}$ not pro-étale covering!

Prop. - Let $\mathcal{F}$ be an abelian étale sheaf on X.
Then the adjunction morphism $\mathcal{F} \to R\nu_* \nu^* \mathcal{F}$ is an isom. In particular $H^i(X_{\acute{e}t}, \mathcal{F}) = H^i(X_{pro\acute{e}t}, \nu^* \mathcal{F})$.

- Back to the prop. (*) (p2):

By assumption, V étale over $\mathbb{T}^n$.

Let
$$\begin{array}{ccc} \tilde{V} & \longrightarrow & \tilde{\mathbb{T}}^n \\ \downarrow & & \downarrow \\ V & \longrightarrow & \mathbb{T}^n \end{array}$$
pro-étale covering of gp $\mathbb{Z}_p^n$.

étale

$\tilde{V} \to \tilde{\mathbb{T}}^n$ is étale, in particular $\tilde{V}$ is perf. aff.

$\tilde{V} = \mathrm{Spa}(S, S^+)$.

Then $\forall i \geq 0$ $H^i(\tilde{V}, \mathcal{O}^+/p) \overset{\text{almost}}{=} \begin{cases} S^+/p & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$

$\Rightarrow H^i(V, \mathcal{O}^+/p) \overset{\text{almost}}{=} H^i(\mathbb{Z}_p^n, S^+/p)$

To need: $\mathrm{im}(H^i(\mathbb{Z}_p^n, S^+/p) \to H^i(\mathbb{Z}_p^n, S'^+/p))$ is almost f.g.

where $\mathrm{Spa}(S', S'^+) = V' \times_V \tilde{V}$.

$\forall \varepsilon > 0$, want to find M_ε f.g. and a map $M_\varepsilon \to \mathrm{im}(-)$ with cokernel killed by $p^?$.

Reduce to check this for $\mathrm{im}\left(H^i((p^m\mathbb{Z}_p)^n, S_m^+/p \otimes_{R_m^+/p} R^+/p\right)$ for $m \geq 0$

$$H^i((p^m\mathbb{Z}_p)^n, R^+/p) \otimes_{R_m^+/p} S_m^+/p \longrightarrow H^i((p^m\mathbb{Z}_p)^n, S'^+_m/p \otimes_{R'^+_m/p} R^+/p))$$

where $R_m^+ = \mathcal{O}_K\langle \underline{T}^{\pm 1/p^m}\rangle$

$R^+ = \mathcal{O}_K\langle \underline{T}^{\pm 1/p^\infty}\rangle$.

- End of the argument:

Def: Let $\hat{\mathcal{O}}^+_{X^b} = \varprojlim_{\varphi} \mathcal{O}_X^+/p$ pro-étale sheaf on X

$\hat{\mathcal{O}}_{X^b} = \hat{\mathcal{O}}^+_{X^b}\left[\frac{1}{p^b}\right]$

Perf_K = category of perfectoid spaces $/K$.

$\to H^i(X_{\text{proét}}, \widehat{\mathcal{O}}^{+}_{X^\flat})$ is a finite dim. $C^\flat$-vs.

Use the exact sequence $0 \to \mathbb{F}_p \to \widehat{\mathcal{O}}_{X^\flat} \to \widehat{\mathcal{O}}_{X^\flat} \to 0$

(to check exactness, reduce to aff. perf.

and use almost purity).

II. The pro-étale and v-topologies on perf. spaces.

Fix a perfectoid field K, with pseudo uniformizer π.

a) Definition.

Def. 1) The big pro-étale site of K is the Grothendieck topo on the category Perf_K for which a family $\{X_i \xrightarrow{f_i} X\}$ is a pro-étale covering if all the f_i are pro-étale and (same cond. as before).

2) The v-topo. is the Grothendieck topo. on Perf_K for which a family $\{X_i \xrightarrow{f_i} X\}$ is a covering if it is a topo. covering + condition.

Rk) If $X \to Y$ qc. morphism of perf. spaces$/K$ and f surjective then $|f|: |X| \to |Y|$ is a quotient map.

(ie $\forall U \subseteq |Y|$ ~~open~~ U open $\Leftrightarrow f^{-1}(U)$ open).

Ex) Let T topological space.

The functor $\underline{T}: \begin{cases} \mathrm{Perf}_K \longrightarrow \mathrm{Sets} \\ X \longmapsto \mathcal{C}^0(|X|, T) \end{cases}$

is a v-sheaf.

b. (Strictly) totally disconnected spaces.

Def.-Prop. Let X be a spectral space

(i.e. q.c. + basis of q.c. open stable by finite intersection and irred. component has a unique generic point).

TFAE: (1) Every open cover (U_i) of X splits

i.e. the map $\sqcup U_i \to X$ splits.

(2) Every connected component of X has a unique closed point.

(3) For any sheaf $\mathcal{F}$ of $\forall i>0$ $H^i(X,\mathcal{F})=0$

If X satisfies this then we say X is totally disconnected.

Def. Let $X \in \mathrm{Perf}_K$. Say that X is totally disconnected if it is qcqs and $|X|$ is totally disconnected.

Rk. Let X be a totally disconnected perfectoid space. There is a continuous map $X \xrightarrow{\pi} \underbrace{\pi_0(X)}_{\text{profinite}}$,

all fibers are of the form $\mathrm{Spa}(L, L^+)$ for some perfectoid field.

Ex. Any such X is affinoid.

Def. Let $X \in \mathrm{Perf}_K$. Say that X is strictly tot. disc. if it is qcqs + any étale cover splits.

Prop. Let X qcqs perf. space. There exists $Y \to X$, surjective proétale (open) morphism, Y str. tot. disconnected

Proof. We can assume X is affinoid perfectoid.

E = system of rep. of isom. classes of $Z \to X$ étale surjective, with Z affinoid perfectoid.

$\forall I \subset E$, let $Z_I = Z_{i_1} \times \dots \times Z_{i_n}$

Let $X_1 = \varprojlim_{I \text{ finite}} Z_I \longrightarrow X_0 = X$ surjective pro-étale aff.

Inductively, define $X_{n+1} \xrightarrow[\text{proétale}]{\text{surj.}} X_n \longrightarrow \dots \longrightarrow X_0 = X$

Then $X_\infty = \varprojlim_n X_n$ is strictly totally disconnected.

Let $Z \to X_\infty$ étale surjective, Z aff. perf.

$\to \exists n$ $W \to X_n$ étale surjective, W aff. perf. (*)

st. $Z = W \times_{X_n} X_\infty$. Then by assumption

$W \times_{X_n} X_{n+1} \to X_{n+1}$ splits.

(*) If $(S_i)_{i\in I}$ cofiltered inverse system of aff. perf. spaces, $S_i = \mathrm{Spa}(R_i, R_i^+)$

$S_\infty = \varprojlim_i S_i$ $\quad S_\infty = \mathrm{Spa}(R_\infty, R_\infty^+)$

$2\text{-}\lim (S_i)_{\text{ét qcqs}} = (S_\infty)_{\text{ét qcqs}}$

* rational subsets : use $|S_\infty| = \varprojlim_i |S_i|$
* finite étale morphism :

$$S_{\infty, \text{fét}} \underset{\substack{\uparrow \\ \text{almost purity}}}{=} (R_\infty^{oa}/\pi)_{\text{fét}} = 2\text{-}\varinjlim (R_i^{oa}/\pi)_{\text{fét}} = 2\text{-}\varinjlim S_{i,\text{fét}}$$

c) Two applications of strictly totally disc. spaces of the study of the v-topo.

Q1 : Is the v-topo ?

ie : is the functor rep. by any $X \in \mathrm{Perf}_K$ is a v-sheaf ?

Prop. $f : Y \longrightarrow X$, X tot. disconnected
$Y = \mathrm{Spa}(S, S^+)$, $X = \mathrm{Spa}(R, R^+)$, $X, Y \in \mathrm{Perf}_K$.

Then $R^+/_\pi \longrightarrow S^+/_\pi$ flat and even faithfully flat if $|f|$ surjective.

Pf: Fact: T profinite topo space.

$\mathcal{A}$ sheaf of rings on T, $\mathcal{M}$ an $\mathcal{A}$-mod on T.

If $\forall t \in T$ $\mathcal{M}_t$ flat $/\mathcal{A}_t$ then $\mathcal{M}(T)$ is flat $/ \mathcal{A}(T)$.

Will apply to:

$Y \xrightarrow{f} X \xrightarrow{pr} \pi_0(X)$

$T = \pi_0(X) \qquad \mathcal{A} = pr_* \mathcal{O}_X^+/\pi$

$\qquad \mathcal{M} = f_* pr_* \mathcal{O}_Y^+/\pi$

Let $t \in T \leftrightarrow X_t$ connected compo. of X
$\qquad \overset{\shortparallel}{\mathrm{Spa}(K_t, K_t^+)} \qquad K_t$ perf. field

Let $\mathrm{Spa}(S_t, S_t^+) = Y \times_X \mathrm{Spa}(K_t, K_t^+)$.

Then $\mathcal{A}_t = K_t^+/_{\pi}$, $\mathcal{M}_t = S_t^+/_{\pi}$

But S_t^+ π-torsion free $\Rightarrow$ S_t^+ flat $/ K_t^+$

By base change $S_t^+/_{\pi}$ flat $/ K_t^+/_{\pi}$.

$\underset{\text{(lemma)}}{\rightarrow}$ $\underset{S^+/\pi}{\overset{\shortparallel}{\mathcal{M}(T)}}$ flat over $\underset{R^+/\pi}{\overset{\shortparallel}{\mathcal{A}(T)}}$ (because no $H^\cdot$ on a tot. disc. space). $\square$

Cor. The presheaves $\mathcal{O}$ and $\mathcal{O}^+$ on Perf_K are v-sheaves.

Pf. First, check that $\mathcal{O}$ and $\mathcal{O}^+$ are pro-étale sheaves.

Can assume X affinoid perfectoid.

We have $\mathcal{O}^{+,a}/\pi_{\text{proét}} = \nu^* \mathcal{O}^{+,a}/\pi_{\text{ét}}$

so it is a sheaf (if $Y = \varprojlim Y_i \to X$

$\mathcal{O}_Y^+(Y)/\pi \overset{\text{almost}}{=} \varinjlim_i \mathcal{O}_{Y_i}^{+a}(Y_i)/\pi$)

The same is true for $\mathcal{O}^+/_{\pi^n}$ $\forall n$.

Taking inverse limit and inverting π, get that $\mathcal{O}$ is a pro-étale sheaf.

Can assume X totally disc., $X = \mathrm{Spa}(R, R^+)$.

Let $f: Y \to X$ v-cover. Can assume that $Y = \mathrm{Spa}(S, S^+)$.

Last prop $\Rightarrow$ $S^+/_{\pi}$ flat over $R^+/_{\pi}$.

So R^+/π is the equalizer of $S^+/\pi \rightrightarrows S^+/\pi \otimes_{R^+/\pi} S^+/\pi$.

But $Y \times_X Y = \mathrm{Spa}(T, T^+)$

$$T^+/\pi \overset{\text{almost}}{=} S^+/\pi \otimes_{R^+/\pi} S^+/\pi$$

$\longrightarrow$ R^+/π is the almost eq. of $(S^+/\pi \rightrightarrows T^+/\pi)$.

The same is true for π^n, $\forall n \geq 1$.

Inverse limit + inverting π, get that $\mathcal{O}$ is a v-sheaf

Q : Is being pro-étale a local property for the proétale topo?

A : No - $\mathbb{D} = \mathrm{Spa}(C\langle T^{\cancel{\pm}1/p^\infty}\rangle, \mathcal{O}_C\langle T^{\cancel{\pm}1/p^\infty}\rangle)$

$f : \mathbb{D} \to \mathbb{D}$ $\quad f^* T^{1/p^n} = T^{2/p^n}$ $\forall n \geq 1$.

f is not proétale ~~loc~~ but $\exists U \to \mathbb{D}$ proétale cover.

such that
$$\begin{tikzcd} V \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & \mathbb{D} \arrow[d, "f"] \\ U \arrow[r] & \mathbb{D} \end{tikzcd}$$
g is proétale.

Thm - Let $\begin{cases} f : Y \to X \text{ qc separated} \\ X \text{ str. tot. disc.} \end{cases}$

quasi proétale $\Big\{$ Assume that for any geom. point $\mathrm{Spa}(C, \mathcal{O}_C) \to X$ then $Y \times_X \mathrm{Spa}(C, \mathcal{O}_C) \to \mathrm{Spa}(C, \mathcal{O}_C)$ is profinite.

Then f is aff. pro étale.

Cor $f : Y \to X$

f proétale locally for proét topo

$\Leftrightarrow$ f quasi proét.