References. P. Scholze Perfectoid Spaces.
J.M. Fontaine Séminaire Bourbaki (2012).

1) Perfectoid fields and algebras.

Def: A perfectoid field is a complete non archimedean field K with residue char $p>0$, whose associated valuation is non discrete and s.t. Frob: $K^\circ/p \to K^\circ/p$ surjective (K° = ring of integers of K).

Ex: $\mathbb{C}_p$ perfectoid field but not $\mathbb{Q}_p$.

Prop: K perfectoid, $\pi \in K$ s.t. $|p| \leq |\pi| < 1$.
$K^{\flat\circ} = \varprojlim_{\text{Frob}} K^\circ/\pi$. Perfect ring of char p.

Multiplicative homeomorphism: $\varprojlim_{x \mapsto x^p} K^\circ \cong K^{\flat\circ}$.

In particular, $K^{\flat\circ}$ indep. of π.
Get a map $\begin{cases} K^{\flat\circ} \to K^\circ \\ x \mapsto x^\# \end{cases}$
There exists $\pi^\flat \in K^{\flat\circ}$ s.t. $|(\pi^\flat)^\#| = |\pi|$
From now on, we assume $\pi = (\pi^\flat)^\#$.
Let $K^\flat = K^{\flat\circ}\left[\frac{1}{\pi^\flat}\right]$. Again $K^\flat \cong \varprojlim_{x \mapsto x^p} K$.
$K^\flat$ is a perfectoid field of char p, complete for the rk 1 valuation defined by $|x|_{K^\flat} = |x^\#|_K$.
Moreover $(K^\flat)^\circ = K^{\flat,\circ}$.

Ex: $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}^{p}$ perfectoid
$K^\flat = \widehat{\mathbb{F}_p((t))[t^{1/p^\infty}]}^{t}$ $\quad t = (p, p^{1/p}, \dots)$

Fix a perfectoid field K.

<u>Def</u> - A perfectoid K-algebra is a Banach K-alg. R s.t. R° = set of power bounded elements in R {is open and bounded}

(R uniform)

$\big(\Leftrightarrow \exists \|.\|$ on R defining its topo. s.t. $\|x^n\| = \|x\|^n$ and $R^\circ = \{x \in R, \|x\| \leq 1\}$ $\big)$.

and $\mathrm{Frob} : R^\circ/p \to R^\circ/p$ surjective.

Exactly as before, we can define the tilt $R^\flat$ of a perfectoid K-alg. R. This is a perfectoid $K^\flat$-alg.

<u>Rk</u>) (1) If R is a perfectoid K-alg,

$$\mathrm{Frob} \quad R^\circ/_{\varpi^{1/p}} \cong R^\circ/_{\varpi}.$$

(2) If R is a $K^\flat$-uniform Banach algebra

R perfectoid $\Leftrightarrow$ R perfect.

<u>Thm</u> - (tilting equivalence)

The tilting functor $R \mapsto R^\flat$ induces an equivalence between $\underbrace{K\text{-Perf}}_{\text{perfectoid } K\text{-alg.}}$ and $K^\flat$-Perf.

<u>Proof</u> - Almost mathematics.

<u>Def</u>: A K°-module M is almost zero if $\mathfrak{m}M = 0$.

The subcategory of K°-mod. fixed by almost zero K°-mod is a Serre category ($\mathfrak{m} = \mathfrak{m}^2$).

Quotient category: $K^{\circ a}$-mod.

Can extent many definitions to the almost category.

<u>Ex</u>) A $K^{\circ a}$-alg., M A-mod. Then M is flat if $X \mapsto M \otimes_A X$ is exact.

If $A = R^a$, $M = N^a$ $\quad M$ A-flat $\Leftrightarrow$ $\forall X$ R-mod $\mathrm{Tor}^R(N,X)$ almost zero

Def 1) A perfectoid K^{oa}-alg. is a π-adically complete flat K^{oa}-alg. A s.t. Frob induces $A/_{\pi^{1/p}} \cong A/_{\pi}$. 2

2) A perfectoid $K^{oa}/_{\pi}$-alg. is a flat $K^{oa}/_{\pi}$-alg. $\overline{A}$ such that $\overline{A}/_{\pi^{1/p}} \cong \overline{A}$

Strategy: $K\text{-Perf} \xrightarrow{①} K^{oa}\text{-Perf} \xrightarrow{②} K^{oa}/_{\pi}\text{-Perf}$

$$\hspace{6cm}\wr\wr$$

$K^\flat\text{-Perf} \xrightarrow{①'} K^{\flat oa}\text{-Perf} \xrightarrow{②'} K^{\flat oa}/_{\pi^\flat}\text{-Perf}$

①, ①' almost mathematics.

②, ②' are equivalences.

Key point: deformation theory: cotangent complex (Gabber-Ramero).

If $\overline{A} \in K^{oa}/_{\pi}\text{-Perf}$ then $\mathbb{L}_{\overline{A}/K^{oa}/\pi} = 0$.

Rem: One can explicitly describe a quasi-inverse of the tilting functor:

Let $R \in K^\flat\text{-Perf}$. Then $R^\# := W(R^o) \otimes_{W(K^{\flat o})} K$

$$\left|\begin{array}{l} W(K^{\flat o}) \xrightarrow{\theta} K \\ \sum_{n\geq 0} [x_n] p^n \mapsto \sum_{n\geq 0} x_n^{\#} p^n \end{array}\right.$$

2) Perfectoid spaces.

Def. A perfectoid affinoid K-alg. is a pair (R, R^+) where $R \in K\text{-Perf}$.

$R^+ \subseteq R^o$ is open and int. closed.

The tilting equivalence extends to perf. aff. alg. ($\mathfrak{m} \subseteq R^+$).

Thm. Let (R, R^+) perf. aff. K-alg., $X = \mathrm{Spa}(R, R^+)$.
Let $(R^\flat, R^{\flat+})$ be the tilting of (R, R^+) $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$.

Then (a) there is a morphism $X \xrightarrow{\sim} X^\flat$ given by mapping $x \in X$ to the valuation $x^\flat \in R^\flat$ given by
$$\forall f \in R^\flat \quad |f(x^\flat)| = |f^\#(x)|$$
This homeomorphism identifies rational subsets
$$\overline{U\left(\frac{f_1, \dots f_n}{g}\right)} = \{x \in X, \ |f_j(x)| \le |g(x)|\}$$
where $f_1, \dots, f_n, g \in R$
$(f_1, \dots, f_n) = R$.

(b) If $U \subseteq X$ rational subset with tilt $U^\flat \subseteq X^\flat$
$(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is a perfectoid affinoid K-alg.
with tilt $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.

(c) The presheaves $\mathcal{O}_X, \mathcal{O}_X^+$ are actually sheaves.

(d) The cohomology groups $H^i(X, \mathcal{O}_X^+)$ is almost zero for all $i > 0$
$(\Rightarrow H^i(X, \mathcal{O}_X) = 0 \ \forall i > 0)$.

Obs: In the def of $U\left(\frac{f_1, \dots, f_n}{g}\right)$, we can always add $f_{n+1} = \pi^N$, $N > 0$
$\left(1 = \sum h_i f_i : h_i \in R \quad \pi^N = \sum \lambda h_i' f_i : h_i' \in R^+ \right)$
$\forall x \in U \ |\pi^N(x)| \le \max |f_i(x)| \le |g(x)|$

Sketch of proof:

(a) $X \to X^\flat$ well defined and continuous
$$\uparrow$$
inverse image of
$$U_{X^\flat}\left(\frac{f_1, \dots, f_n}{g}\right) \text{ is } U\left(\frac{f_1^\#, \dots, f_n^\#}{g}\right)$$

* Check directly : if $U = U\left(\frac{f_1^\#, \dots, f_n^\#}{g^\#}\right)$

then $\mathcal{O}_X(U)$ is a perfectoid K-alg. with tilt $\mathcal{O}_{X^\flat}(U^\flat)$.
and $\mathcal{O}_X(U)^\circ \underset{\text{almost}}{=} R^\circ \left\langle \left(\frac{f_1^\#}{g^\#}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\#}{g^\#}\right)^{1/p^\infty} \right\rangle$

* The pbm is that $\begin{cases} R^\flat \to R \\ f \mapsto f^\# \end{cases}$ is not surjective.

To deal with this, need to prove that you can "approxi-

From that, get $X^{\flat} \cong X^{\flat}$.

(c)+(d): Enough to prove that for any covering (U_i) of X by rational subsets, then each $H^m(C^\cdot)$ are almost zero

where $C^\cdot = (\mathcal{O}_X^+(X) \to \bigoplus_i \mathcal{O}_X^+(U_i) \to \bigoplus_{i,j} \mathcal{O}_X^+(U_i \cap U_j) \to \dots)$

$\rightsquigarrow$ ($\mathcal{O}_X$ sheaf ($\Rightarrow \mathcal{O}_X^+$ sheaf)
higher coho. of $\mathcal{O}_X^+$ almost vanishes.

Step 1 Reduce to $X^{\flat}$.

Assume that you know that the statement is true for $X^{\flat}$.

We have $C^\cdot/_{\pi} = C^{\flat\cdot}/_{\pi^{\flat}}$.

where $C^{\flat\cdot}$ is the Čech complex attached to the covering $(U_i^{\flat})$ of $X^{\flat}$

$\longrightarrow H^m\left(C^\cdot/_{\pi}\right)$ is ~~m~~ $\mathfrak{m}$-torsion.

$\underset{\mathfrak{m}^2 = \mathfrak{m}}{\Longrightarrow}$ $H^m(C^\cdot/_{\pi^m})$ —— $\forall m \geq 1$

$\Rightarrow$ $H^m(C^\cdot) = H^m(\varprojlim_m C^\cdot/_{\pi^m})$ is killed by $\mathfrak{m}$.

Step 2 Noetherian approximation.

Need that $(\pi^{\flat})^{1/m}$ kills $H^m(C^\cdot)$ $\forall m$.

Assume that $K = \widehat{\mathbb{F}_p[t^{1/p^\infty}]}^t \left[\frac{1}{t}\right]$.

Prop. There exists a filtered inductive system of f.t. reduced $\mathbb{F}_p[t]$-alg. (S_j) s.t. $R^+ = \left(\varinjlim_j (S_j)^{\wedge t}_{\mathrm{perf}}\right)^{\wedge t}$

Upshot: $C^\cdot = \left(\varinjlim_j (C_j^\cdot)^{\wedge t}_{\mathrm{perf}}\right)^{\wedge t}$

where C_j is the Čech complex of a rigid space over $\mathbb{F}_p((t))$ for some rational covering.

Step 3 Conclusion.

Claim: $\forall n\ \forall j$ $H^n(C_j^\cdot)$ killed by t^N for some $N>0$.

Proof = Tate acyclicity $\Rightarrow C_j^\cdot\left[\frac{1}{t}\right]$ exact.

Banach approximation mapping:

$t^N \ker d^n \subseteq \operatorname{Im} d^{n-1}$ for some $N>0$. $\square$

Corollary: t^{1/p^m} kills $H^n(C^\cdot)$ for any m and n.

Proof t^{1/p^m} kills $H^n((C_j^\cdot)_{perf}) = \varinjlim_{\varphi} H^n(C_j^\cdot)$. $\square$

Rem. For a given affinoid rigid space, you can't find a uniform N as in the claim.

Def A perfectoid space$/K$ is an adic space$/K$ locally isomorphic to $\operatorname{Spa}(R,R^+)$, (R,R^+) perfectoid affinoid K-alg.

Cor Tilting gives an equivalence between the category of perfectoid spaces$/K$ and perfect spaces$/K^\flat$

$X \longmapsto X^\flat$ for ~~even~~ any perf. aff. K-alg (R,R^+)

$\operatorname{Hom}(\operatorname{Spa}(R,R^+),X) = X(R,R^+) = X^\flat(X^\flat, X^{\flat +})$

X and $X^\flat$ are homeomorphic, $U \subseteq X$ perf. aff.

$\Leftrightarrow U^\flat \subseteq X^\flat$ perf. aff.

and $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ tilts to $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.

3) The almost purity theorem.

Def. (i) A morphism $(R,R^+) \to (S,S^+)$ of perf. aff. K-alg is finite étale if $R \to S$ is finite étale and $S^+ =$ int. closure of R^+ in S.

(ii) $f: X \to Y$ morphism of perfectoid spaces$/K$ is finite étale if $\exists (V_i)$ covering $V_i = \operatorname{Spa}(R_i, R_i^+)$ perf. aff.

s.t. $U_i = f^{-1}(V_i) = \operatorname{Spa}(S_i, S_i^+)$ perf. aff.

(iii) $f : X \to Y$ étale if there is a covering (U_i) of X s.t. $f|_{U_i}$ = open immersion composed with a finite étale morphism.

<u>Thm</u>. Let X be a perfectoid space $/K$.
The tilting functor induces an equivalence between $X_{ét}$ and $X^\flat_{ét}$.

Let $X = \mathrm{Spa}(R, R^+)$.

$$\begin{array}{ccccccc}
R_{fét} & \xleftrightarrow{①} & R^{oa}_{fét} & \cong & R^{oa}/\pi\text{-}fét & \cong & R^{\flat oa}_{fét} \xrightarrow{②} R^\flat_{fét} \\
| & & & & & & \\
K\text{-}\mathrm{Perf} & \xrightarrow{\sim} & K^{oa}\text{-}\mathrm{Perf} & \xrightarrow{\sim} & K^{oa}/_{\pi}\text{-}\mathrm{Perf} & \xrightarrow{\sim} & K^{\flat oa}\text{-}\mathrm{Perf} \xrightarrow{\sim} K^\flat\text{-}\mathrm{Perf}
\end{array}$$

To prove the thm, it is enough to see that ① and ② are equivalences. In other words, you want:

<u>Thm</u>. Let $R \in K\text{-}\mathrm{Perf}$ and $R \to S$ finite étale.
Then S is perfectoid and $R^{oa} \to S^{oa}$ is finite étale.

<u>Proof</u>. <u>Step 1</u>: ② is an equivalence (so the thm is true in char p)

<u>Step 2</u>: Prove that ① is an equivalence when $R = L$ is a perfectoid field.

<u>Step 3</u>: Let $X = \mathrm{Spa}(R, R^+)$, $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat +})$.
Let S as in the thm. For any rational subset $U \subseteq X$, $S(U) = S \otimes_R \mathcal{O}_X(U)$. This is a finite étale $\mathcal{O}_X(U)$-covering.
Let $x \in X$. We have:

$$2\text{-}\varinjlim_{U \ni x} \mathcal{O}_X(U)_{fét} \cong \widehat{k(x)}_{fét}$$

Indeed $2\text{-}\varinjlim \mathcal{O}_X(U)_{fét} = \mathcal{O}_{X,x,fét}$
$\varinjlim \mathcal{O}_X(U)$ is Henselian so $\mathcal{O}_{X,x,fét} = \left(\widehat{\mathcal{O}^+_{X,x}\left[\frac{1}{\pi}\right]}\right)_{fét}$
$= \widehat{k(x)}_{fét}$

Cor. As tilting identifies residue fields,

$$\ell\text{-}\varinjlim_{x\in U} \mathcal{O}_X(U)_{f\acute{e}t} \cong \widehat{k(x)}_{f\acute{e}t}$$

$$\underset{\text{Step 2}}{\cong} \widehat{k(x)}^{\flat}_{f\acute{e}t}$$

$$= \ell\text{-}\varinjlim_{U^\flat \ni x^\flat} \mathcal{O}_{X^\flat}(U^\flat)_{f\acute{e}t}$$

$\Rightarrow$ You can find a covering (U_i) of X s.t. $S(U)$ is the space of global sections of a finite étale covering V_i of U_i s.t. $S_i^{\circ a}$ is finite étale $/R_i^{\circ a}$.

$\overset{\shortparallel}{\mathrm{Spa}}(S_i, S_i^+)$ " $\mathrm{Spa}(R_i, R_i^+)$

(Step 1)

The V_i will glue to a finite étale covering Y of X with $S'^{\circ a}$ is finite étale over $R^{\circ a}$. $\overset{\shortparallel}{\mathrm{Spa}}(S', S'^+)$

By the sheaf property, we necessarily have that $\mathcal{O}(Y) = S$

Thm. For any perfectoid space X over K, $\mathcal{O}_X, \mathcal{O}_X^+$ is an étale sheaf and if X is perf. aff.

~~then~~ $H^i(X_{\acute{e}t}, \mathcal{O}_X^+) \overset{\text{almost}}{=} 0 \quad \forall i > 0$.

Proof: Again, you need to check that the coho. of the Čech complex attached to an étale covering of X, given as rational subsets of finite étale coverings of rational subsets of X, is almost zero.

Tilt everything to char p. Reduce to prove the same for $X^\flat / K^\flat$. Then use noetherian approximation

4) Application: the direct summand conjecture.

Hochster (André, Bhatt)

Thm. Let R regular ring. Let $R \to S$ any $\overset{\text{injective}}{\text{morphism}}$ which makes S a finitely presented R-module.

Then $R \to S$ splits in R-mod.

Observations: (5)

i) If $R \to S \to S'$ and if $R \to S'$ splits in Mod_R, then $R \to S$ splits.

ii) Reduce to the case where R is $\widehat{W(k)[x_1, \dots x_d]}^p$, k perfect field of char p. (a regular local ring, res. char p, p complete, p-torsion free.)

Strategy of the proof:

Choose $g \in R$ prime to p s.t. $R\left[\frac{1}{pg}\right] \to S\left[\frac{1}{pg}\right]$ is finite étale

Will construct $R \to R_\infty$ s.t.

* $\left(R_\infty\left[\frac{1}{p}\right], R_\infty\right)$ is a perf. aff. K-alg. where $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$
* $R \to R_\infty$ is almost faithfully flat.
* g admits $p^{n\text{th}}$-roots in R_∞, for all n.

Assume R_∞ constructed.

Consider the exact sequence $0 \to R \to S \to S/R \to 0$.

Define $\alpha \in \mathrm{Ext}^1_R(S/R, R)$. Want $\alpha = 0$.

Base change to R_∞:

$$\begin{array}{ccccc} R & \longrightarrow & S & & \\ \downarrow & & \downarrow & & \\ R_\infty & \longrightarrow & S'_\infty & \longrightarrow & S_\infty = \text{int. closure of } S'_\infty \text{ in } S_\infty\left[\frac{1}{p}\right] \end{array}$$

Claim: It is enough to show that $R_\infty \to S'_\infty$ is almost split w.r.t. (pg)

$\Leftarrow$ $R_\infty \to S_\infty$ almost split.

(ie $\forall n > 0$ $\exists f_n : S'_\infty \to R_\infty$ s.t. coker and ker of $R_\infty \to S'_\infty \xrightarrow{f_n} R_\infty$ minus id are killed by $(pg)^{1/p^n}$).

(Pf: uses $R \to R_\infty$ faithfully flat + Krull intersection thm).

Consider $\forall n$ $R_\infty\left\langle \frac{p^n}{g} \right\rangle \xrightarrow{\varphi_n} R_\infty\left\langle \frac{p^n}{g}\right\rangle \otimes_{R_\infty} S_\infty$

$\varphi_n\left[\frac{1}{p}\right]$ is finite étale and so by almost purity,

$$R_{\infty,n} := R_\infty\left\langle \frac{p^n}{g}\right\rangle \to R_\infty\left\langle \frac{p^n}{g}\right\rangle \otimes_{R_\infty} S_\infty =: S_{\infty,n}$$

is almost finite étale.

This implies that the map φ_n almost splits.

($A \hookrightarrow B$ finite étale, $\mathrm{tr}: B \to A$ surjective,

can find $b \in B$, $\mathrm{tr}\, b = 1$. Splitting : $x \in B \mapsto \mathrm{tr}(bx)$).

To conclude, need to prove:

$$\mathrm{Ext}^1_{R_\infty}(S_\infty/R_\infty, R_\infty) \underset{\text{almost}}{=} \varprojlim_n \mathrm{Ext}^1_{R_{\infty,n}}(S_{\infty,n}/R_{\infty,n}, R_{\infty,n})$$

To construct R_∞, s.t. $(R_\infty[\frac{1}{p}], R_\infty)$ perf. aff. Viale

* construct $R \to R'_\infty$ ~~[illegible]~~ almost faithfully flat.
* consider the perfectoid closed disk over R'_∞ :

$$Y = \mathrm{Spa}(R'_\infty\langle T^{1/p^\infty}\rangle\left[\tfrac{1}{p}\right], R_\infty\langle T^{1/p^\infty}\rangle).$$

Let Z be the Zariski closed subspace def by $(T - g)$

$\quad\ \ \| $

$\mathrm{Spa}\left(R_\infty\left[\frac{1}{p}\right], R_\infty\right)$.

Need $R'_\infty \to R_\infty$ almost faithfully flat.

(I ideal of Y, Z defined by Y, $Z = \bigcap_{f_1,\dots,f_n \in I} U_{f_1,\dots f_n}$).