07/06/2018 Last time: ①

LL3

- $SL_2/\mathbb{R}$ $\phi_k : W_\mathbb{R} \to PGL_2(\mathbb{C})$

  G^* $\phi_k(z) = \begin{bmatrix} (z/\bar z)^k & 0 \\ 0 & 1 \end{bmatrix}$, $\phi(j) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

  $|\Pi_\varphi| = 2$, $|S_\varphi| = 2$, char id ✓

- $G = D^1$ $|\Pi_\varphi| = 1$, $|S_\varphi| = 2$ char id ✓ up to sign

- $G^* = SL_2/\mathbb{Q}_p$ $\phi : W \to PGL_2(\mathbb{C})$

  $\theta : E^1/(1+\mathfrak{p}_E)^1 \to \mathbb{C}^\times$ E/F unramif. quad.

  $1 \to E^\times \to W_{E/F} \to \Gamma \to 1$

  $\phi(z) = \begin{bmatrix} \theta(z/\bar z) & 0 \\ 0 & 1 \end{bmatrix}$, $\phi(j) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

  $\theta \neq \mathrm{sgn}$ $|\Pi_\phi| = 2$, $|S_\phi| = 2$, char id ✓

  $\theta = \mathrm{sgn}$ $|\Pi_\phi| = 4$, $|S_\phi| = 4$, char id ✓

- $G = D^1$ $\theta \neq \mathrm{sgn}$ $|\Pi_\phi| = 2$, $|S_\phi| = 2$ char id ✓ up to sign

  $\theta = \mathrm{sgn}$ $|\Pi_\phi| = 1$, $|S_\phi| = 4$ char id X

Problems when G not q-split:

- misual $|\Pi_\phi|$ and $|\mathrm{Irr}(\pi_0(\overline{S_\phi}))|$
- no natural normalization of LS transfer factor

  LLS define $\Delta : H(F) \times G(F) \to \mathbb{C}$

  well defined up to scalar in S^1

  when G q-split, Whittaker data fixes an iso.
- Arthur: any normalization of TF is not invariant under iso of end. data.
- Failure of char id. for any normalization.

Inner forms:

G conn. red. /F

exists G^* q-split /F $\xi : G^* \xrightarrow[/\bar F]{\sim} G$ s.t. for all $\sigma \in \Gamma$ $\xi^{-1}\sigma(\xi) \in \mathrm{Inn}(G^*)$.

G^* is uniquely def. by G, ξ is not.

$G^* \rightsquigarrow$ Choose (T^*, B^*) Borel pair $/\bar{F}$

$\rightsquigarrow$ ~~based~~ root datum $\circlearrowleft \Gamma$

G general: Any two Borel pairs (T_1, B_1), (T_2, B_2) are canonically iso, $Ad(g)$ for any $g \in G(\bar{F})$ st. $Ad(g) : (T_1, B_1) \to (T_2, B_2)$.

Take limit $\rightsquigarrow$ abstract Borel pair $(T_0, B_0)/\bar{F}$

$\rightsquigarrow$ based root datum $\circlearrowleft \Gamma$

Given $\xi : G^* \to G$, then $\widehat{G^*} = \widehat{G}$, ${}^LG^* = {}^LG$ so can use ${}^LG^*$ instead of ${}^LG$.

<u>Crude LLC</u>: Still make sense for G except

$$\begin{tikzcd} LL : \Pi(G) \arrow[r] \arrow[dr, two heads] & \Phi(G) \\ & \Phi_{rel}(G) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

need not to be surj. ϕ relevant for G if the unique levi ${}^LM \subset {}^LG$ through which ϕ factors universally, ~~[illegible]~~ is the L-group of a levi of G.

<u>Vogan's idea</u>:

<u>Ex</u> Unitary groups: $\phi : W_{\mathbb{R}} \to {}^LU_n$ discrete

Have the unitary groups $U_{p,q}$, $p+q = n$, all inner to each other.

$U_{p,q} = U_{q,p}$.

$|\overline{S_\phi}| = 2^{n-1}$, $|\Pi_\phi(U_{p,q})| = \binom{n}{p} = \binom{n}{q}$

Q-split: $U_{\frac{n}{2},\frac{n}{2}}$, $U_{\frac{n+1}{2},\frac{n-1}{2}}$

$$\sum_{p+q=n} \binom{n}{p} = 2^n, \quad |S_\phi| = 2^n \qquad S_\phi = Cent(\phi, \widehat{G})$$

$$\overline{S_\phi} = S_\phi(Z(\widehat{G}))^\Gamma$$

$$\begin{tikzcd} H^1(\Gamma, \overset{U_n^*}{\overset{\shortparallel}{G}}) \arrow[r, two heads] \arrow[d, phantom, "\shortparallel"] & H^1(\Gamma, G^*_{ad}) \arrow[d, phantom, "\shortparallel"] \\ \{p+q=n\} & \{p+q=n\} / (p,q) \leftrightarrow (q,p) \end{tikzcd}$$

<u>Conj</u>: there exists a bijection

$$\begin{tikzcd}
\bigsqcup_{u \in H^1(\Gamma, G)} \Pi_\phi(G^u) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Irr}(\pi_0(S_\phi)) \arrow[d] \\
H^1(\Gamma, G) \arrow[r, "\text{Kottwitz}"'] & \pi_0(Z(\hat{G})^\Gamma)^*
\end{tikzcd}$$

Kottwitz : $(H, \mathcal{S}, \quad)$ ext. endo. triple

$\gamma \in H(F)_{rs}$, $\delta \in G(F)_{rs}$

$\updownarrow$ there is an inv. measuring rel. position

$\delta^* \in G^*(F)_{rs}$

. $\Delta[\mathfrak{w}, e](\gamma, \delta^*)_{(\xi, z)}$ $\quad z \in Z^1(\Gamma, G^*)$

$\xi^{-1}\sigma(\xi) = z_\sigma \in Z^1(\Gamma, G^*_{ad})$

Modify ξ s.t. $\xi(\delta^*) = \delta$

$\mathrm{inv}(\delta^*, \delta) = z \in Z^1(\Gamma, T_{\delta^*})$

$e(G^z)$ Kott. sign . $\Delta[\mathfrak{w}, e, z](\gamma, \delta) = \Delta[\mathfrak{w}, e](\gamma, \delta^*)\langle \mathrm{inv}(\delta^*, \delta), s_{\gamma\delta^*} \rangle$.

<u>Conj</u> : $\sum_{\pi \in \Pi_\phi(G)} \langle s, \pi \rangle \Theta_\pi(\delta) = \sum_{\gamma \in H(F)/st} \Delta[\mathfrak{w}, e, z](\gamma, \delta)$

<u>Issue</u>: $H^1(\Gamma, G^*) \longrightarrow H^1(\Gamma, G^*_{ad})$ not surjective in general.

<u>Ex</u> $G^* = GL_n, SL_n, GSp_{2n}, Sp_{2n} \ldots$

Over $\mathbb{R}$: ABV propose a solution ad hoc:

replace $Z^1(\Gamma, G^*)$ with pre-image in $C^1(\Gamma, G^*)$ of $Z^1(\Gamma, G^*_{ad})$.

<u>Kottwitz's set $B(G)$</u>:

$F/\mathbb{Q}_p$, $L = \breve{F}$ completion of max unramified extension.

σ Frobenius

$B(G) = G(L)/_{\sigma\text{-conj.}}$, $x \mapsto g x \sigma(g)^{-1}$.

$$B(G) = H^1(\mathbb{Z}, G(L)) \xrightarrow[\text{bij.}]{} H^1(W_F, G(\bar{L})).$$

Recall: An isocrystal is a finite dim L-vs V with σ-linear $f : V \to V$.

Fix $b \in G(L)$. For any $(\rho, V) \in \mathrm{Rep}_F(G)$.

$(V \otimes_F L, b \circ (1 \otimes \sigma))$ is an isocrystal.

Its iso. class depends only on $[G] \in B(G)$.

Get $B(G) \to \mathrm{Func}(\mathrm{Rep}_F(G), \mathrm{Isoc}_F)$

Exercise: What is the image?

Isoc_F is Tannakian with an obvious fiber functor

$$\mathrm{Isoc}_F \to \mathrm{Vect}_L .$$

It is s.s. with simple objects given by $r/s \in \mathbb{Q}$

$$V_{r/s} = \left(L^{\oplus s}, \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ \pi^r & & & 0 \end{bmatrix} \cdot \sigma \right)$$

π unif.

$$V = \bigoplus_{r/s} V_{r/s}^{m(r/s)} \quad \text{slope decompo.}$$

V basic / isoclinic / isotypic if $m(r/s) \neq 0$ for unique r/s.

Galois (Langlands - Rapoport): F is of char 0

Def A Galois gerbe is an ext.

$$\begin{tikzcd} 1 \arrow[r] & H(\overline{F}) \arrow[r] & \mathcal{E} \arrow[r] & \Gamma \arrow[r] \arrow[l, bend left, dashed, "s"] & 1 \end{tikzcd}$$

H is an aff. alg group def over $\overline{F}$ st. there is a section over $\Gamma_E \subseteq \Gamma_F$ for a finite ext. E/F with $\mathrm{Ad}(s(\sigma))$ giving H a ratio. structure over E.

Ex: If H is def $/F$, $H(\overline{F}) \rtimes \Gamma$ called neutral.

If V a finite dim. $\overline{F}$-vs

$$\mathcal{GL}(V) = \{ (f, \sigma) \mid \sigma \in \Gamma, \ f : V \xrightarrow[\sigma\text{-lin}]{} V \}$$

$$1 \to GL(V) \to \mathcal{GL}(V) \to \Gamma \to 1 .$$

Def. A homo. of Galois g. is a homo. / Γ algebraic on the kernels.

Def. A rep. of a Galois gerbe in the $\overline{F}$ vs V is a homo to $GL(V)$.

Construction. Given $(\mathcal{C}, \omega)$ $\mathcal{C}$ Tannakian cat., ω a fiber functor / $\overline{F}$, associate

$$\mathcal{E} = \{(f, \sigma) \mid \sigma \in \Gamma,\ f : \sigma^* \circ \omega \xrightarrow{\sim} \omega\}$$

σ^* endo functor on $\mathrm{Vect}_{\overline{F}}$, $\sigma(V) = V \otimes_{\overline{F}, \sigma} \overline{F}$.

Explicitly $f = (f_c)_{c \in \mathrm{Ob}(\mathcal{C})}$ $\quad f_c : \omega(c) \to \omega(c)$ σ-linear

The fiber functor ω tautologically becomes

$$\omega : \mathcal{C} \to \mathrm{Rep}(\mathcal{E}).$$

This is an equivalence of cat.

Apply this to $B(G)$:

The cat. Isoc_F with tautological fiber functor is assigned the gerbe

$$1 \to \mathbb{D} \to \mathcal{E} \to \Gamma \to 1$$

where $\mathbb{D} = \varprojlim \mathbb{G}_m$ is $X^*(\mathbb{D}) = \mathbb{Q}$ with trivial Γ-action.

The gerbe for $\mathrm{Rep}_{\overline{F}}(G)$ is $G(\overline{F}) \rtimes \Gamma$.

Every $b \in \underset{G(L)}{\cancel{B(G)}}$ leads to

$$\begin{tikzcd} \mathcal{E} \arrow[rr] \arrow[dr] & & G(\overline{F}) \rtimes \Gamma \arrow[dl] \\ & \Gamma & \end{tikzcd}$$

i.e. to an element of $H^1_{\mathrm{alg}}(\mathcal{E}, G(\overline{F}))$.

Upshot: $B(G) = H^1_{\mathrm{alg}}(\mathcal{E}, G(\overline{F}))$.

Invariants of $[b] \in B(G)$:

Restrict b to $\mathbb{D}$, get $1 \to H^1(\Gamma, G) \to H^1_{alg}(\mathcal{E}, G)$

$$\to \left(\frac{Hom(\mathbb{D}, G)}{G}\right)^{\Gamma}$$

Take $\nu : \mathbb{D} \to G$. Choose Borel pair (T^*, B^*) in G^*, $\nu^* = \xi^{-1} \circ \nu : \mathbb{D} \to T^*$.

Now $\nu^* \in Hom(X^*(T), \mathbb{Q}) = X_*(T^*) \otimes \mathbb{Q}$.

$$(X_*(A) \otimes \mathbb{Q})^+ = (X_*(T^*)^\Gamma \otimes \mathbb{Q})^+ \subseteq (X_*(T^*) \otimes \mathbb{Q})^+$$

$A \subseteq T^*$ split subtorus.

$$H^1(\Gamma, G) \hookrightarrow B(G)_{basic} \hookrightarrow B(G)$$

$$\nu : \mathbb{D} \to Z_G \subseteq G$$

K-map:

$$\begin{tikzcd} B(G) \arrow[r, "K"] & \pi_1(G)_\Gamma = X^*(Z(\hat{G})^\Gamma) \\ B(G)_{basic} \arrow[u, hook] \arrow[ur, "\sim"] & \end{tikzcd}$$

Pair $(\nu^*_{[b]}, K(b))$ determines $[b]$ uniquely.

$$\begin{tikzcd} B(G^*) \arrow[r] & B(G^*_{ad}) \\ B(G)_{basic} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & B(G^*_{ad})_{basic} = H^1(\Gamma, G^*_{ad}) \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

$b \in B(G^*)_{basic} \rightsquigarrow$ inner form of G^*

If $b \in B(G^*)$ then $Cent(\nu^*_{[b]}, G^*)$ is a Levi $M^* \subseteq G^*$ and $[b] \rightsquigarrow$ inner form of M^*.

Conj.

$$\begin{tikzcd} \bigcup_{[b] \in B(G)_{basic}} \Pi_\phi(G) \arrow[r, "\sim"] \arrow[d] & Irr_{alg}(S_\varphi / (S_\varphi \cap \hat{G}_{der})^0) \arrow[d] \\ B(G)_{basic} \arrow[r, "K"] & X^*(Z(\hat{G})^\Gamma) \end{tikzcd}$$

G

+ char ids (for ϕ tempered).

- If Z_G is connected, $B(G^*)_{basic} \twoheadrightarrow H^1(\Gamma, G^*_{ad})$
- If Z_G is finite, $H^1(\Gamma, G^*) \hookrightarrow B(G^*)_{basic}$.

Let F be of char 0.

Consider $u = \varprojlim_{\substack{E/F,\, N \in \mathbb{N} \\ \text{finite Gal.}}} \operatorname{Res}_{E/F} \mu_N/\mu_N$

Rmk: $H^1_{cts}(\Gamma, u) = 0$

$$H^2_{cts}(\Gamma, u) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & F = \mathbb{R} \\ \widehat{\mathbb{Z}}, & F/\mathbb{Q}_p \end{cases}$$

Prop. The functorial map (Z finite multiplicative)

$\operatorname{Hom}_F(u, Z) \longrightarrow H^2(\Gamma, Z)$ is surjective.

It is also injective if Z splits.

Consider $1 \to u \to \mathcal{E} \to \Gamma \to 1$.

Functor $[Z \to G]$, G aff. alg., $Z \subseteq G$ finite

$$\downarrow$$

$$H^1(u \to \mathcal{E}, Z \to G) \subseteq H^1(\mathcal{E}, G).$$

$$H^1_{basic}(\mathcal{E}, G) = \varinjlim_{Z \in Z(G)} H^1(u \to \mathcal{E}, Z \to G).$$

Rmk. $H^1_{basic}(\mathcal{E}, G) \to H^1(\Gamma, G_{ad})$ is always surj.

$\overline{G} = G/Z$,

$$\begin{tikzcd}
 & \overline{G}(F) \arrow[r, phantom, "="] \arrow[d] & \overline{G}(F) \arrow[d] & & \\
1 \arrow[r] & H^1(\Gamma, Z) \arrow[r] \arrow[d] & H^1(u \to \mathcal{E}, Z \to Z) \arrow[r] \arrow[d] & \operatorname{Hom}_F(u, Z) \arrow[r] \arrow[d, equal] & * \\
1 \arrow[r] & H^1(\Gamma, G) \arrow[r] \arrow[d, equal] & H^1(u \to \mathcal{E}, Z \to G) \arrow[r] \arrow[d] & \operatorname{Hom}_F(u, Z) \arrow[r] \arrow[d] & * \\
 & H^1(\Gamma, G) \arrow[r] & H^1(\Gamma, \overline{G}) \arrow[r] \arrow[d] & H^2(\Gamma, Z) \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}$$

Thm. [illegible] There is a unique functorial homo

$$H^1(u \to \mathcal{E}, Z \to G) \longrightarrow \pi_0(Z(\widehat{G})^+)^* \text{ st}$$

1) its restriction to $[1 \to T]$, T torus coincides with Tate-Nakayama, and

2) is compatible $\pi_0(Z(\bar{G})^+) \to \mathrm{Hom}_F(u, Z)$.

Rmk - . Compatible with $H^1(\Gamma, G) \to \pi_0(Z(\hat{\bar{G}})^\Gamma)^*$.

. Reduces to tori

$$1 \to H^1(\Gamma, T) \to H^1(u \to \mathcal{E}, Z \to T) \to \mathrm{Hom}_F(u, Z).$$

This is bijective when $F/\mathbb{Q}_p$ or G is a torus, otherwise has explicit description of failure.

[illegible] : $F = \mathbb{R}$, $H^1_{basic}(\mathcal{E}, G)$ canonically iso to

Thm ABV strong real torus.

(rig) Conjecture -

$$\begin{tikzcd}
\bigcup\limits_{h \in H^1(u \to \mathcal{E}, Z \to G^*)} \Pi_\varphi(G^h) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Irr}(\pi_0(S_\varphi^+)) \arrow[d] \\
H^1(u \to \mathcal{E}, Z \to G^*) \arrow[r] & \pi_0(Z(\hat{\bar{G}})^+)^*
\end{tikzcd}$$

where

$$\begin{tikzcd}
\hat{\bar{G}} \arrow[r] & \hat{G} \\
S_\varphi^+ \arrow[r] \arrow[u, phantom, sloped, "\subseteq"] & S_\varphi \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

Moreover, can use h to normalize TF.

$$e(G^h) \sum_{\pi \in \Pi_\varphi(G^h)} \langle \dot{s}, \pi \rangle \Theta_\pi(\delta) \text{ [illegible]}$$

$$= \sum_{\gamma \in H(F)/st} \Delta[\,](\gamma, \delta)\, S\Theta_\varphi(\gamma)$$

Forces to refine the notions of

1) endoscopic data

2) isomorphism of endoscopic data.

Known cases: ⑤

1) $F = \mathbb{R}$, conjecture holds.

2) $F/\mathbb{Q}_p$: G splits over tame ext.

$p \nmid \#$ Weyl group

φ supercuspidal parameter.

<u>Drawback</u>: No linear alg. description of this gerbe.

Comparison between $B(G)$ and $H^1(\mu \to \mathcal{E}, \_)$:

$\mathrm{Hom}_F(\mu, Z) \longrightarrow H^2(\Gamma, Z)$

$\mu := \varprojlim \mu_N \hookrightarrow \mathbb{D} = \varprojlim \mathbb{G}_m$

$\mathrm{Hom}_F(\mu, \mu) \longrightarrow H^2(\Gamma, \mu) \longrightarrow H^2(\Gamma, \mathbb{D})$

$$\begin{tikzcd}
1 \arrow[r] & \mu \arrow[r] \arrow[d] & \mathcal{E}^{rig} \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{D} \arrow[r] & \mathcal{E}^{Kot} \arrow[r] & \Gamma \arrow[r] & 1
\end{tikzcd}$$

$$\begin{tikzcd}
B(G)_{basic} \arrow[r] \arrow[d, "\kappa"] & H^1_{basic}(\mathcal{E}^{rig}, G) \arrow[d] \\
X^*(Z(\hat{G})^\Gamma) \arrow[r] & \pi_0(Z(\hat{G})^+)^*
\end{tikzcd}$$

Given ϕ $\quad \pi_0(S_\phi^+) \longrightarrow S_\phi$

<u>Thm</u> - 1) ϕ discrete, we get a commutative diagram:

$$\begin{tikzcd}
\mathrm{Irr}_{alg}(S_\varphi) \arrow[r] \arrow[d] & \mathrm{Irr}(\pi_0(S_\varphi^+)) \arrow[d] \\
B(G^*)_{basic} \arrow[r] & H^1_{basic}(\mathcal{E}, G)
\end{tikzcd}$$

Top. map is an iso. on fibers.

2) Conj. (rig) true for all G^* with conn. center iff (rig) is true for all G^* without assumption on center.

3) Conj. (iso) true " "' iff. (rig) is true " " "'.