04/06/2018 LLC (1)

$F/\mathbb{Q}_p$ finite ext., G qs. conn. red. gp $/F$

${}^LG \cong \widehat{G} \rtimes \begin{cases} \mathrm{Gal}_F \\ W_F \end{cases}$

conn. red. $/\mathbb{C}$ (→ $\widehat{G}$)

Three cases of functoriality:

1) M Levi subgp of G (= centralizer of some split torus in G).
$\iota_M : {}^LM \hookrightarrow {}^LG$ (up to $\widehat{G}$-conj)

2) $G \to H$ with normal image, ${}^LH \to {}^LG$.

3) T/F max. torus of G, $\widehat{T} \cong T \leftarrow$ max. torus.
Gal. act. differ by a 1-cocycle in Weyl.

Langlands parameters:

Two variants:

1) $\phi' : W_F' \longrightarrow {}^LG$, $\phi' : W_F \longrightarrow {}^LG$ continuous (commuting with the projections to W_F)

$N \in \widehat{\mathfrak{g}} = \mathrm{Lie}\,\widehat{G}$ s.t. $\mathrm{Ad}(\phi'(w))(N) = |w| N$.
$\forall w \in W_F \setminus I_F$ $\phi'(w)$ ss.

2) $\phi : WD_F \longrightarrow {}^LG$ (commuting with the projections to W_F) continuous on W_F, alg. on SL_2, ss.

. Prop. ϕ as above, the Levi subgp $\iota_M({}^LM)$ of ${}^LG$ minimally containing $\phi(WD_F)$ form one $\widehat{G}$-conj. class.

. Langlands classification (see Silberger-Zink):
apply polar decomposition to $\phi(w)$ $w \in W_F \setminus I_F$
$\leadsto$ get canonical $\phi = \phi_0 \chi$
ϕ_0 is bounded (its proj. of $\phi(W_F)$ to $\widehat{G}$ is bounded) taking values in some ${}^LM$.
$\chi : W_F \to X_*(Z(\widehat{M})^{\mathrm{Gal},\circ}) \otimes_{\mathbb{Z}} \mathbb{R}_{>0}$ unr.

Essentially bdd parameters $\Phi_{temp}(G) \subset \Phi(G)$
(Φ_{temp}: bdd mod $Z(\widehat{G})$; Φ: all ss ϕ)

Discrete ϕ : those such that ${}^L\Pi$ in $\mathcal{B}_{\text{rop}} = {}^LG$

$\Leftrightarrow S_\phi = \text{Cent}(\phi, \widehat{G}) / Z(\widehat{G})^{\text{Gal}}$ finite.

$\longrightarrow$ notation: $\Phi_{\text{disc}}(G)$.

Conj (crude local Langlands):

$\exists$ surj. finite to 1 map $LL : \Pi(G) \longrightarrow \Phi(G)$

irr. rep. of $G(F)$

such that: (denoting $LL^{-1}(\phi) = \Pi_\phi(G)$):

1] ϕ is essentially $\begin{cases} \text{bounded} \\ \text{tempered} \end{cases}$

$\Leftrightarrow$ for one (all) $\pi \in \Pi_\phi(G)$ is (are) ess. tempered.

2] ϕ is discr. $\Leftrightarrow$ for one (all) $\pi \in \Pi_\phi(G)$ is (are) ess. L^2.

3] If ϕ is discr. and $\phi|_{SL_2}$ is trivial,

($\Leftrightarrow$) any $\pi \in \Pi_\phi(G)$ is supercuspidal.

4] If M Levi subgp of G and $\phi_M \in \Phi(M)$ is bounded (not just modulo $Z(\widehat{M})$) then

$$\Pi_{\iota_M \circ \phi_M}(G) = \{ \text{constituents of } i_P^G(\pi_M) \text{ for } \pi_M \in \Pi_{\phi_M}(M) \}.$$

5] LL is compatible with twists and central char.:

i) Z = conn. center of G

$p_Z : {}^LG \longrightarrow {}^LZ$, $\forall \pi \in \Pi_\phi(G)$ $\quad \omega_\pi : Z(F) \longrightarrow \mathbb{C}^\times$

$LL(p_Z \circ \phi)$

ii) $\exists$ natural $H^1(W_F, Z(\widehat{G})) \longrightarrow \text{Hom}_{\text{cont}}(G(F), \mathbb{C}^\times)$.

Expect compatibility with twisting.

6] M Levi subgp of G, $\pi_M \in \Pi(M)$ then

$\forall$ subquotient π in $i_P^G(\pi_M)$,

$LL(\pi) \circ \iota_W \cong \iota_M \circ LL(\pi_M) \circ \iota_W \quad , \iota_W : W_F \hookrightarrow W_F \times SL_2(\mathbb{C})$.

7) If $G \to H$ has normal image and abelian coker $\forall \pi \in \Pi(H)$ and any direct summand π' of $\pi|_{G(F)}$ (ss. and f.l.) $LL(\pi') = LL(\pi)$ composed with ${}^LH \to {}^LG$.

L and ε-factors conj. attached to $\pi \in \Pi_?(G)$ and $r: {}^LG \to GL(V)$, but not enough to characterize ϕ
($\exists G, \phi_1, \phi_2$ s.t. $\forall r$ $r \circ \phi_1 \cong r \circ \phi_2$ but $\phi_1 \not\cong \phi_2$).

For most applications, need more refined conj.

Recall:

Thm (Harish-Chandra) $\forall \pi \in \Pi(G)$, the distribution
$\Theta_\pi : f \in \mathcal{C}_c^\infty(G(F)) \mapsto tr(\pi(f))$.
is represented by $\Theta_\pi \in L^1_{loc}(G(F))$
(sm. on $G(F)_{reg}$ ($\leftarrow$ reg. ss. elt. in $G(F)$).
i.e. $tr(\pi(f)) = \int_{G(F)} f(g)\, \Theta_\pi(g)\, dg$.

$\Theta_\pi|_{G(F)_{reg}}$ characterizes π
is inv under $G(F)$-conj.
$\leftrightarrow$ sm. functions $(T(F)_{reg} \xrightarrow{sm.} \mathbb{C})_{T \text{ max tori of } G / G(F)\text{-conj.}}$
inv. under $N_{G(F)}(T)/T(F)$.

Refined LLC for qs G:
$S_\phi = Cent(\phi, \hat{G})$, $\overline{S_\phi} = Cent(\phi, \hat{G}) / Z(\hat{G})^{Gal}$.
(may be non abelian)

Conj. $\forall \phi \in \Phi(G)$, should have bij.
$$Irr(\pi_0(\overline{S_\phi})) \simeq \Pi_\phi(G)$$
(reduced to ess. temp. case).

To make bij. canonical, need to fix a Whittaker datum $\mathfrak{w} = (B, \theta)$ for G

B: Borel subgp $/F$; θ: generic unitary char. of $U(F)$, $U = R_u(B)$.

Conj. (Shahidi) $\forall \phi \in \Phi_{temp}(G)$ $\exists! \pi \in \Pi_\phi(G)$ which is $\mathfrak{w}$-generic. The bij:

$$L_{\mathfrak{w}} : \mathrm{Irr}(\pi_0(\overline{S_\phi})) \cong \Pi_\phi(G)$$

$$1 \leftrightarrow \text{generic } \pi.$$

$s \in S_\phi$ ss.

Define
$$\Theta_\phi^{s,\mathfrak{w}} = \sum_{\pi \in \Pi_\phi(G)} \underbrace{\langle s, \pi \rangle_{\mathfrak{w}}}_{tr(L_{\mathfrak{w}}^{-1}(\pi)(s))} \Theta_\pi$$

and $S\Theta_\phi := \Theta_\phi^{1,\mathfrak{w}}$.

Consider $\mathrm{Cent}(s, \widehat{G})^\circ := \widehat{H}$. Get $\mathcal{H} = \widehat{H}\,\phi(W_F) \subset {}^LG$

$$1 \to \widehat{H} \to \mathcal{H} \to W_F \to 1$$

In g^{al} $\nexists$ section fixing a pinning of $\widehat{H}$. Assume that $\exists$ such a section, so that $\eta : \mathcal{H} \xrightarrow{\sim} {}^LH$.

$\mathfrak{e} = (H, s, \eta)$ is called an extended endoscopic triple.

Also get $\phi_H \in \Phi_{temp}(H)$ st. $\phi = \eta^{-1} \circ \phi_H$.

Expect that $\Theta_\phi^{s,\mathfrak{w}}$ and $S\Theta_{\phi_H}$ are related by a certain kernel function:

transfer factors $\Delta[\mathfrak{e}, \mathfrak{w}](.,.)$.

Support of $\Delta[\mathfrak{e}, \mathfrak{w}]$:

correspondence between stable ss(G)-reg conj. classes in $H(F)$

ss-reg. conj. classes in $G(F)$.

Stable conjugacy:

$\delta, \delta' \in G(F)$ ss. (strongly reg.) are stable by conj. if $\exists g \in G(\overline{F})$ s.t. $g \delta g^{-1} = \delta'$

If $T = \mathrm{Cent}(\delta, G)$ $\quad \{\delta' \text{ st conj. to } \delta\}/_{G(F)\text{-conj}}$

$$\cong \ker\left(H^1(F,T) \to H^1(F,G)\right).$$

<u>Thm</u> (Kottwitz). $H^1(F,G) \cong \pi_0\left(Z(\hat{G})^{\mathrm{Gal}}\right)^\vee$

functorial in $T \underset{\max}{\hookrightarrow} G$ and in $G \to H$ having normal image.

$[\gamma]$ G-reg ss. stable conj. class in $H(F)$

$\rightsquigarrow$ a stable ss. reg. conj. class in $G(F)$

= union of finitely many conj. classes $[\delta]$.

$\exists$ natural iso. $\underset{\mathrm{Cent}(\gamma, H)^\circ}{T_H} \cong \underset{\mathrm{Cent}(\delta, G)^\circ}{T}$

$s \in Z(\hat{H})^{\mathrm{Gal}_F} \hookrightarrow \hat{T}_H^{\mathrm{Gal}_F} \cong \hat{T}^{\mathrm{Gal}_F} \ni s_{\gamma,\delta}$

Get pairing between $\{[\delta'] \sim [\delta]\}$ and $\hat{T}^{\mathrm{Gal}_F}/Z(\hat{G})^{\mathrm{Gal}_F}$.

Property of transfer factors:

$$\Delta[w, z](\gamma, \delta') = \Delta[w, z](\gamma, \delta) \langle \mathrm{inv}(\delta, \delta'), s_{\gamma,\delta}\rangle^{-1}$$

<u>Conj.</u> (refined LLC for qs G).

1) $\forall \phi \in \Phi_{\mathrm{temp}}(G)$ $\quad S\Theta_\phi$ is stable

(is inv under stable conjugacy).

2) $\forall s \in S_\phi$ s.s. $\quad \Theta_\phi^{s,w}(\delta) = \sum_{\gamma \in H(F)/\mathrm{st\,conj}} \Delta[w,z](\gamma,\delta) S\Theta_{\phi_H}(\gamma)$

$\forall \delta \in G(F)$ ss. reg.

<u>Rmk</u>. This determines ι_w provided it exists.

Also, one can check (validity for one w

$\Leftrightarrow$ validity for all w.

Example of SL_2:

• $F = \mathbb{R}$

$SL_2/\mathbb{R}$, $\quad T \hookrightarrow SL_2 \hookleftarrow S$

$T(\mathbb{R}) = \mathbb{R}^\times \qquad S(\mathbb{R}) = S^1.$

$W(S,G) = N(S,G)/S$

$W(S,G)(\mathbb{R}) \curvearrowright S^1$ via inversion

Discrete parameters:

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{\mathbb{C}/\mathbb{R}} \longrightarrow \Gamma_{\mathbb{R}} \longrightarrow 1$$

$z \in \mathbb{C}^\times$, $j \in W_{\mathbb{C}/\mathbb{R}} \longmapsto \sigma \in \Gamma_{\mathbb{R}}$, complex conj.

$$\begin{cases} j z j^{-1} = \bar{z} \\ j^2 = -1 \end{cases}$$

Any discrete par $\phi : W_{\mathbb{C}/\mathbb{R}} \to PGL_2(\mathbb{C})$ is equivalent to $\phi(z) = \begin{bmatrix} (z/\bar{z})^k & 0 \\ 0 & 1 \end{bmatrix}$ for some $k > 0$

$$\phi(j) = \begin{bmatrix} 0 & (-1)^k \\ 1 & 0 \end{bmatrix}$$

$$S_\phi = \begin{bmatrix} \pm 1 & 0 \\ 0 & 1 \end{bmatrix} \simeq \mathbb{Z}/2\mathbb{Z}.$$

$$\Pi_\phi(G) = \{ D_k^+, D_k^- \}$$

Discrete series of wt k: $\quad \Theta_k^+(y) = \dfrac{-y^k}{y - y^{-1}}$, $y \in S^1$

$$\Theta_k^-(y) = \frac{y^{-k}}{y - y^{-1}}, \quad y \in S^1$$

$$\Theta_k^{\pm}(x) = \begin{cases} \dfrac{x^{-k}}{x - x^{-1}}, & x > 1 \\[2ex] \dfrac{x^k}{x - x^{-1}}, & x < 1 \end{cases}$$

$$S\Theta_\phi = \Theta_k^+ + \Theta_k^-$$

$$S\Theta_\phi(y) = \frac{y^k - y^{-k}}{y - y^{-1}}$$

$$S\Theta_\phi(x) = 2 \cdot \begin{cases} \cdots \\ \cdots \end{cases}$$

$\Theta_\phi^{s,w} = \Theta_k^+ - \Theta_k^-$, $\Theta_\phi^{s,w}(y) = -\dfrac{y^k + y^{-k}}{y - y^{-1}}$

$\Theta_\phi^{s,w}(x) = 0.$

$\hat{H} = \mathrm{Cent}(s, \hat{G})^\circ = \begin{bmatrix} * & \\ & * \end{bmatrix} \cong \mathbb{C}^\times.$

H is a 1-dim. $\mathbb{R}$-torus.

$\mathfrak{H} \cong {}^L H$ precisely when $H = S$ anisotropique.

$\mathbb{C}^\times \rtimes \Gamma = {}^L H \xrightarrow{L\eta} \mathrm{PGL}_2(\mathbb{C})$

$z \rtimes 1 \longmapsto \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$

$1 \rtimes \sigma \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

ϕ factors as $L\eta \circ \phi_H$ for $\phi_H : W_{\mathbb{C}/\mathbb{R}} \longrightarrow {}^L H$.

$z \longmapsto (z/\bar{z})^k \rtimes 1$, $j \longmapsto (-1)^k \rtimes \sigma$.

$\xrightarrow{\text{via LLC for tori}}$ $\chi_k : S^1 \longrightarrow \mathbb{C}^\times$, $y \longmapsto y^k$.

$\Theta_\phi^{s,w}(y) = \dfrac{\mathrm{sgn}(y - y^{-1})}{|y - y^{-1}|} \cdot (\underbrace{y^k}_{\chi_k(y)} + \underbrace{y^{-k}}_{\chi_k(y^{-1})})$

$\Delta(y^\pm, y) = \dfrac{\mathrm{sgn}(y - y^{-1})}{|y - y^{-1}|}.$

There are two Whittaker data:

Fix $B = \left(\nabla \right)$, $N = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

$\Psi : \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \longmapsto e^{\pm ix}.$

D_k^+ generic for e^{ix}, D_k^- is generic for e^{-ix}.

• $F/\mathbb{Q}_p$: ~~[scribbled out]~~

$p>2$

Let $E|F$ unramif. quad. ext.

$E^1 = \ker(N: E^\times \to F^\times)$

Take char $\theta: E^1 \to \mathbb{C}^\times$ trivial on $(1+\mathfrak{p}_E)^1$.

θ gives a char $\bar\theta$ on $k_E^1 \subset SL_2(k_F)$.

Deligne-Lusztig induction gives a cuspidal rep. $R_{\bar\theta}$ of $SL_2(k_F)$. $SL_2(F)$ has two hyperspecials:

$$SL_2(\mathcal{O}_F) \text{ and } \begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix} SL_2(\mathcal{O}_F) \begin{pmatrix}\pi^{-1} & 0\\ 0 & 1\end{pmatrix}.$$

$\leadsto$ get $R_{\theta,1}$, $R_{\theta,2}$ supercuspidal.

There are 4 Whittaker data:

If $\Lambda: F \to \mathbb{C}^\times$ non trivial, $\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix} \mapsto \Lambda(\varepsilon . x)$

$$\varepsilon \in F^\times / F^{\times,2} = \{1, \eta, \pi, \eta\pi\}$$

(unif. π; η = lift of gen. of $k_F^\times$)

If $\theta \neq \theta^{-1}$, ie $\theta \neq \mathrm{sgn}$, then $R_{\bar\theta}$ irred.

so $R_{\theta,1}$ and $R_{\theta,2}$ are irred. and form $\Pi_\phi(G)$.

$R_{\theta,1}$ gen for $\varepsilon = 1, \eta$

$R_{\theta,2}$ gen for $\varepsilon = \pi, \eta\pi$.

If $\theta = \theta^{-1}$, then $R_{\bar\theta}$ reducible into 2 pieces.

$\leadsto$ get 4 supercuspidals reps. in $\Pi_\phi(G)$.

First case: $S_\phi \cong \mathbb{Z}/2\mathbb{Z}$ just like over $\mathbb{R}$

Second —: $S_\phi = \left\{\begin{pmatrix}1 & \\ & 1\end{pmatrix}, \begin{pmatrix}-1 & \\ & 1\end{pmatrix}, \begin{pmatrix} & 1\\ 1 & \end{pmatrix}, \begin{pmatrix} & -1\\ 1 & \end{pmatrix}\right\}$

$$\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

For $y \in k_E^1 \setminus \{\pm 1\} \in E^1 \setminus \{\pm 1\}$

$$S\Theta_\phi(y) = -\frac{\Theta(y) + \Theta(y^{-1})}{|y - y^{-1}|}.$$

There are two tori in $SL_2(F)$ stably conj ...

SL_2 has a unique inner form G

$G(F) = D^1$, D/F quat. alg., D^1 = reduced norm = 1.

<u>$F = \mathbb{R}$</u> $G(F) = D^1$ compact Lie group.

Rat. conj. = stable conj.

$\phi_k \leftrightarrow$ singleton L-packet containing unique of dim $k+1$.

$\{ |S_\phi| = 2, \quad |\Pi_\phi(G)| = 1, \quad S\Theta_\phi(y) = \frac{y^k - y^{-k}}{y - y^{-1}}$.

$s = 1$: $H = G^* = SL_2/\mathbb{R}$

$$S\Theta_{\varphi,G}(y) \longleftrightarrow S\Theta_{\varphi,G^*}(y)$$

$$\frac{y^k - y^{-k}}{y - y^{-1}} \qquad\qquad -\frac{y^k - y^{-k}}{y - y^{-1}}$$

$s \neq 1$: $\Theta^s_{\varphi,G} = \frac{y^k - y^{-k}}{y - y^{-1}}$

$$= - \frac{sgn(y - y^{-1})}{|y - y^{-1}|} (y^k - y^{-k})$$

$$\Delta(y, y^{\pm}) = \frac{sgn(y - y^{-1})}{|y - y^{-1}|}.$$

<u>$F / \mathbb{Q}_p$</u> : $G(F) = D^1$

Rat. conj. $\neq$ stable conj.

There is a unique rat. conj. class of tori S with $S(F) = E^1$, but Weyl group realizes out stable conj.

G has a unique ~~pro~~ parahoric sub, namely $G(F)$ itself. Reductive quotient is k_E^1

$\Theta : E^1 / (1 + \mathfrak{p}_E)^1 \longrightarrow \mathbb{C}^\times$ gives char $\bar{\Theta} : k_E^1 \to \mathbb{C}^\times$.

But k_E^1 is the red. quotient of $G(F)$.

$\Theta \neq \Theta^{-1}$ then both Θ, Θ^{-1} give ~~two~~ uniq reps,

L-packet has size 2. $\leftrightarrow$ 2 on G^*

$\Theta = \Theta^{-1}$ then Θ gives only one rep.,

L-packet has size 1 $\leftrightarrow$ 4 on G^*