30/05/2018 The Local Langlands Correspondence. ①

<u>Goal</u>: formulate LLC for general G over p-adic field F and illustrate.

1) "Basic" representation theory of reductive groups.

F as above, G/F conn. red. group $(GL_n, Sp_{2n} \dots)$

C: coefficient field $(= \mathbb{C}$ or $\overline{\mathbb{Q}_\ell})$.

Notion of smooth and admissible rep. of $G(F)$:

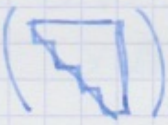
(π, V) $\quad \pi : G(F) \longrightarrow GL(V)$ $\quad V$ vs $/C$

π is <u>smooth</u> if $\forall v \in V$ $\exists$ open subgroup $K \subset G(F)$ s.t. K fixes v

<u>admissible</u> if $\forall K$ open compact subgroup, $\dim_C V^K < \infty$.

π smooth + irreducible $\Rightarrow$ adm.

Parabolic induction: $P = MN$ parabolic subgp of G (← unip. rad. over N)

($G = GL_n$ $\quad P =$  $\quad M \cong GL_{n_1} \times \dots \times GL_{n_r}$)

π smooth rep. of $M(F)$

$i_P^G(\pi)$ normalized parabolic induction of π (need to choose $\sqrt{q}$ in C).

r_P^G adjoint functor

<u>Fact</u>: i_P^G, r_P^G are exact, preserve finite length

π rep. of $M(F)$ has f.l. $\Rightarrow$ so does $i_P^G(\pi)$.

<u>Notation</u>: for GL_n: if $\pi \cong \pi_1 \otimes \dots \otimes \pi_r$, π_i smooth rep. of GL_{n_i}

$$i_P^{GL_n}(\pi) = \pi_1 \times \dots \times \pi_r .$$

<u>Def</u>: π smooth irr. rep. of $G(F)$ is <u>supercuspidal</u> if $\forall P \subsetneq G$ $\; r_P^G(\pi) = 0$.

Fact $\Leftrightarrow$ all coefficients are compactly supported mod center.
$$g \mapsto \langle \pi(g) v, v \rangle.$$

Fact π smooth rep. of $M(F)$ has f.l.,
$i_P^G(\pi)^{ss}$ is independent of the choice of P
(eg. $G = GL_n$, may permute the GL_{n_i}'s).

Thm (Jacquet) Supercuspidal support:
$\forall \pi$ smooth irr. rep. of $G(F)$, $\exists P = MN$,
σ smooth irr. supercuspidal repr. of $M(F)$
s.t. $\pi \hookrightarrow i_P^G(\sigma)$.
If (M', σ') s.t. π occurs in $i_{P'}^G(\sigma')^{ss}$
supercuspidal then $(M, \sigma) \underset{G(F)\text{-conj}}{\sim} (M', \sigma')$.

Bernstein center:
(M, σ) as above, variety $\Theta_{M,\sigma}$ of unr. twists of σ
($\sigma \otimes \chi$, $\chi : M(F) \to \mathbb{C}^\times$ in the image of $\underset{M \to GL_1}{\underline{\mathrm{Rat}(M)}} \otimes_{\mathbb{Z}} \mathbb{C}^\times$)
(only a finite subgp of such χ is stabilizing σ).

$W(\Theta_{M,\sigma})$ = stab. of $\{\sigma \otimes \chi\}_\chi$ under
$W_M = N_{G(F)}(M) / M(F)$ finite group.

$\Theta_{M,\sigma} / W(\Theta_{M,\sigma})$ $\mathbb{C}$-points of a (possibly singular) variety $/\mathbb{C}$.

$\to$ Bernstein center: $\mathbb{C}$-alg. of endo. of the identity functor on cat. of all smooth rep of $G(F)$.

Thm (Bernstein, Deligne)
$$\text{Bernstein center} \cong \varprojlim_{K \text{ open cpt}} Z(C_c^\infty(K \backslash G(F) / K, \mathbb{C}))$$
$$\text{also} \cong \prod_{(M,\sigma)/\text{unramified twists}/G(F)\text{-conj.}} (\text{reg. functions on } \Theta_{M,\sigma} / W(\Theta_{M,\sigma})).$$

Now take $C = \mathbb{C}$: (2)

Notions of • essentially square-integrable irr. rep. of $G(F)$ (all coeff. are $L^2 / Z(G(F))$). Characterized in terms of r_P^G.

• essentially tempered rep: weaker growth condition on coeff. Also characterized in terms of r_P^G.

Supercuspidal $\Rightarrow$ ess. L^2 $\Rightarrow$ ess. temp. $\Rightarrow$ unitary.

Prop. Any tempered rep. π of $G(F)$ occurs as a direct summand of $i_P^G(\sigma)$, for σ ess. L^2 with unitary central char. Pair (Π, σ) is well-defined up to $G(F)$-conj.

2) Weil (-Deligne) groups.

$F/\mathbb{Q}_p$ finite, $Gal_F := Gal(\overline{F}/F)$, k: residue field

$$\begin{array}{ccccccccc} 1 & \to & I_F & \to & Gal_F & \to & Gal_k \cong \hat{\mathbb{Z}} & \to & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \to & I_F & \to & W_F & \to & \mathbb{Z} & \to & 1 \end{array}$$

topology: I_F is open, endowed with its induced topo.

3 versions of the Weil-Deligne group: $|w| = q^{-v(w)}$

1) $W_F' = \mathbb{G}_a \rtimes W_F$ where action is $w x w^{-1} = |w| x$
over $\mathbb{Q}$ or C

2) $WD_F = W_F \times SL_2$ alg. gp/$\mathbb{Q}$ or C

3) $W_F \times SU(2)$

Prop: V finite dim. v.s./C

(W_F') $\rho : W_F \to GL(V)$ cont. (for discrete topo on V)

$N \in \mathrm{End}\, V$ s.t. $\rho(w) N \rho(w)^{-1} = |w| N$

($\Rightarrow$ N nilpotent). Frob-ss: $\forall w \in W_F \setminus I_F$ $\rho(w)$ is ss.

WD_F : rep. alg. on Sl_2.

Choose $\sqrt{q}$ in $\mathbb{C}$. $i_w : \begin{cases} W_F \hookrightarrow W_F \times Sl_2(\mathbb{C}) \\ w \mapsto (w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})) \end{cases}$

Start with $\varphi : W_F \times Sl_2 \to Gl(V)$.

Consider $\left(\varphi \circ i_w, d\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$ is a rep. of W'_F.

<u>Fact</u>. Up to equivalence, get same notions of Weil-Deligne repr. (Jacobson-Morozov).

<u>LL for Gl_1</u>:

actually $W_F = \varprojlim_{E/F \text{ finite Galois}} W_{E/F}$

$$1 \to E^\times \to W_{E/F} \to Gal(E/F) \to 1.$$

In particular, $W_F \to \varprojlim_{E/F} Gal(E/F) = Gal_F$

also have $(E = F)$ $W_F \to F^\times$ inducing $W_F^{ab} \xrightarrow{\sim} F^\times$.

In particular :

$$LL \quad Hom_{cont}(F^\times, \underset{\text{discrete topo.}}{\mathbb{C}^\times}) \xrightarrow{\sim} Hom_{cont}(W_F, \mathbb{C}^\times).$$

3) <u>Gl_n</u> :

$U : \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ (0) & & 1 \end{pmatrix} \right\} \subseteq Gl_n$

Fix $\Psi : F \to \mathbb{C}^*$ cont. non trivial.

$u \in U(F)$, $\Theta(u) := \Psi(u_{1,2} + \ldots + u_{n-1,n})$

cont. "generic" char. on $U(F)$.

<u>Def</u>: (π, V) irr. rep. smooth of $G(F)$ is generic if $\exists \lambda_{\neq 0} : V \to \mathbb{C}$ s.t. $\forall v \in V$, $u \in U(F)$,

$$\lambda(\pi(u)v) = \Theta(u)\lambda(v).$$

Then $\pi \hookrightarrow \left\{ \text{Whittaker functionals } g \mapsto \lambda(\pi(g)v) \right\}_{v \in V}$

Thm (Bernstein - Zelevinsky).

1) Any ess. square int. ir. rep. π of $GL_n(F)$ can be realized as the unique ir. subrep. of

$$\sigma|\det|^{\frac{d-1}{2}} \times \dots \times \sigma|\det|^{\frac{1-d}{2}}$$

σ supercusp. rep. of $GL_m(F)$, $md = n$

The pair (σ, d) is determined by π.

2) For $\sigma_1, \dots, \sigma_r$ unitary ess. L^2 ir. rep. of $GL_{n_i}(F)$, $\sigma_1 \times \dots \times \sigma_r$ is ir. - In particular, any ess. temp. rep. of $GL_n(F)$ is generic.

Analogous classification Galois side:

Any ir. rep. of WD_F is $\cong \rho \otimes \nu_d$ (ρ ir rep of W_F, ν_d ir rep of SL_2 of dim d.)

Any ss. rep. of WD_F is $\bigoplus_i \rho_i \otimes \nu_d$
(bounded ones on W_F: each ρ_i is unitary.)-

Langlands classification: any smooth ir rep. of $GL_n(F)$ is $\cong$ unique ir. quotient of $\sigma_1|\det|^{t_1} \times \dots \times \sigma_r|\det|^{t_r}$
with σ_i temp. and $t_1 > \dots > t_r$

Ex) $\underset{GL_1}{|.|^{\frac{d-1}{2}}} \times \dots \times \underset{GL_1}{|.|^{\frac{1-d}{2}}} \longrightarrow \mathrm{triv}_{GL_n(F)}$

Thm (Harris - Taylor, Henniart, Scholze).

$\exists!$ unique LLC for $GL_n(F)$, that is a family of bij. LL between smooth ir rep. of $GL_n(F)$ (n varies) and ss. cont. $WD_F \longrightarrow GL_n(\mathbb{C})$ s.t.

1) For $n = 1$, recover LL.

2) If $LL(\pi) = \rho$ then $LL(\omega_\pi) = \det \rho$
ω_π central char. $GL_1(F) \hookrightarrow GL_n(F)$

and for any $\chi \in \mathrm{Hom}_{cont}(F^\times, \mathbb{C}^\times)$, $LL(\pi \otimes (\chi \circ \det)) = LL(\pi) \otimes LL(\chi)$.

3) $LL(\widetilde{\pi}) = LL(\pi)^*$.

4) Compatibility with L- and ε-factors:

$L(\pi_1 \times \pi_2, s) = L(LL(\pi_1) \otimes LL(\pi_2), s)$

$\varepsilon(\pi_1 \times \pi_2, s, \psi) = \varepsilon(LL(\pi_1) \otimes LL(\pi_2), s, \psi)$.

Def of L- and ε-factors:

(ρ, N) rep. of W_F' $\quad L((\rho,N), s) = \det(1 - q^{-s}\mathrm{Frob}|(\ker N)^{I_F})$

ε-factor for (ρ, N) : complicated.

- Tate's thesis for $n = 1$ (char. of $W_F \cong F^\times$)
- Only an existence thm (Langlands, Deligne) global proof

$\exists!$ family $(\varepsilon(\rho, s, \psi))_{F,\rho}$

which is compatible with char., additive and inductive on Weil rep. of dim 0 ...

On the $GL_n(F)$ side:

L functors and ε-functors defined by Jacquet, Piatetski-Shapiro and Shalika

"integrals pairing Whittaker functionals"

Fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ generated by $L(\pi_1 \times \pi_2, s)$.

If π_1, π_2 supercusp. with $n_1 \neq n_2$,

$$L(\pi_1 \times \pi_2, s) = 1.$$

Rks:

1) Henniart and Harris-Taylor used "numerical local Langlands". Not Scholze's proof.

2) Three proofs use the Arthur-Clozel BC (analogue of $\mathrm{Res}^{W_F}_{W_E}(-)$ on the rep. of $GL_n(F)$ side). Proof is global using twisted trace formula.

$\Rightarrow$ LL is compatible with BC.

3) Harris-Taylor and Scholze: essential use of Shimura varieties $\rightsquigarrow$ characterization of LL in terms of local-global compatibility (4)

π ess. L^2 irrep of $GL_n(F)$. Can embed it in π^{glob} aut. cusp. rep. of $GL_n/_{E^{glob}}$

E^{glob} CM field
F^{glob} totally real.

(v of F^{glob} split in E^{glob} with $(F^{glob})_v \xrightarrow{\sim} F$.

- π^{glob} is conj. self dual.
- At arch. places, π^{glob} "algebraic regular"
- $\pi^{glob}_v \cong \pi$ up to unr. twist
- \+ other conditions.

For such π^{glob}'s, have cont. ss. l-adic rep. ($l \neq$ res. char of F)

$$\rho : Gal_{E^{glob}} \longrightarrow GL_n(\overline{\mathbb{Q}_l}).$$

Local-global compat. : $\rho|_{Gal_F} \simeq LL(\pi \otimes (\text{unr twist})) \otimes (\text{additional twist})$

4) Scholze's characterization :

$\forall h \in \mathcal{C}_c^\infty(GL_n(F))$, $\forall \tau \in W_F$ mapping to $\mathbb{Z}_{>0}$,

$\exists f_{\tau,h} \in \mathcal{C}_c^\infty(GL_n(F))$ s.t. $\forall \pi$ irrep. of $GL_n(F)$

$$tr(f_{\tau,h} | \pi) = tr(\tau | LL(\pi)(\text{twist}))\, tr(h|\pi)$$

$f_{\tau,h}$ constructed "geometrically" and locally.

4) Local Langlands for Tori.

T torus $/F$, $X^*(T)$: gp of char

$T_{\overline{F}} \longrightarrow GL_{1,\overline{F}}$, free $\mathbb{Z}$-module with cont. Gal_F-action

$X^*(T) \times X_*(T) \longrightarrow \mathbb{Z}$ perf. pairing.

Note: for any E/F splitting T,

$$T(E) \cong E^\times \otimes_{\mathbb{Z}} X_*(T).$$

$$\hat{T} := \mathbb{C}^\times \otimes_{\mathbb{Z}} X^*(T) \quad (\text{torus over } \mathbb{C}) \quad \circlearrowleft \mathrm{Gal}_F$$

Thm (Langlands) $\exists$ isom. LL:

$$\mathrm{Hom}_{cont}(T(F), \mathbb{C}^\times) \xrightarrow{\sim} H^1_{cont}(W_F, \hat{T}).$$

Construction:

Natural morphism

$$\mathrm{Hom}_{cont}(E^\times, \hat{T})_{\mathrm{Gal}(E/F)} \xrightarrow{\text{cor.}} H^1_{cont}(W_{E/F}, \hat{T}) = H^1_{cont}(W_F, \hat{T})$$

Non-trivial part: this is an iso.

(follows from a Tate-Nakayama like iso

+ partial resolution of T : $T \hookrightarrow T'$ induced from E/F)

$$\mathrm{Hom}_{cont}(E^\times, \hat{T}) \cong \mathrm{Hom}_{cont}(E^\times, X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times) = \mathrm{Hom}_{cont}(\underbrace{E^\times \otimes_{\mathbb{Z}} X_*(T)}_{T(E)}, \mathbb{C}^\times)$$

Have natural $\mathrm{Hom}_{cont}(T(E), \mathbb{C}^\times)_{\mathrm{Gal}(E/F)} \xrightarrow{\text{res}} \mathrm{Hom}_{cont}(T(F), \mathbb{C}^\times)$ iso (using $T \hookrightarrow T'$ as above).

5) The dual group.

Rmk. $H^1_{cont}(W_F, \hat{T})$

$$= \left\{ \varphi : \begin{tikzcd} W_F \arrow[r] \arrow[dr] & \hat{T} \rtimes \mathrm{Gal}_F \arrow[d] \\ & \mathrm{Gal}_F \end{tikzcd}, \text{ cont., compatible with } \to \mathrm{Gal}_F \right\} \Big/ \hat{T}\text{-conj.}$$

Recall triple (G, B, T) G connected red. gp $/F$

B Borel subgp —

T max torus in B $/$ iso

$1\text{-}1 \updownarrow$

Based root data $(X, R, \Delta, Y, R^\vee, \Delta^\vee) \circlearrowleft \mathrm{Gal}_F$

$X = X^*(T)$, roots, simple roots, $Y = X_*(T)$ $/$iso

G s.t. $\exists (B,T)$ is called quasi-split.

$\nwarrow$ unique up to $G(F)$-conj.

For $\widehat{G}$ = conn. red. over $\mathbb{C}$ $\leftrightarrow (Y, R^\vee, \Delta^\vee, X, R, \Delta)$.

Fix $(\mathcal{B}, \mathcal{T})$ Borel pair in $\widehat{G}$.

$+ \ (X_\alpha)_{\alpha \in \Delta^\vee}$ in $\mathrm{Lie}(R_u(\mathcal{B}))$.

$\rightsquigarrow$ split exact sequence

$$1 \to \widehat{G}_{ad} \to \mathrm{Aut}(\widehat{G}) \to \mathrm{Out}(\widehat{G}) \to 1$$

$$\widehat{G}_{ad} = \widehat{G}/Z(\widehat{G}), \qquad \mathrm{Out}(\widehat{G}) = \mathrm{Aut}(\text{based root data})$$

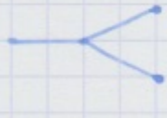
<u>Def</u>: ${}^L G = \widehat{G} \rtimes \begin{cases} \mathrm{Gal}_F \\ W_F \end{cases}$

<u>Ex</u>: 1) $G = GL_n$, $\widehat{G} = GL_n(\mathbb{C})$ with trivial Gal_F-action

2) $G = SO_{2n+1}$, $\widehat{G} = Sp_{2n}(\mathbb{C})$ ———

3) Non-trivial Galois action:

type D_4



D_n

$\mathrm{Gal}_F \to S_3$ $\mathrm{Gal}_F \to \mathbb{Z}/2$

Four cases of functoriality:

1) (G, B, T) as before, $\widehat{T} \xrightarrow{\sim} \mathcal{T}$ extends to

${}^L T \xrightarrow{\sim} \mathcal{T} \rtimes \mathrm{Gal}_F$.

2) T maximal torus of G defined over F,

choose B Borel$/\overline{F}$ $\supseteq T_{\overline{F}}$ $\rightsquigarrow$ get $\widehat{T} \xrightarrow{\sim} \mathcal{T}$

but Galois actions differ by 1-cocycle taking values in $W(T_{\bar{F}}, G_{\bar{F}}) \simeq W(\mathcal{T}, \hat{G})$.

3) $G \to H$ morphism whose image is a non-trivial subgp. get ${}^L H \to {}^L G$.

6) $\left\{ \begin{array}{l} \text{Parabolic} \\ \text{Levi} \end{array} \right.$ - subgps of ${}^L G$:

$P|_F$ parabolic subgp of G (can assume $P \supset B$) / $G(F)$ conj.

$\updownarrow$ 1-1

${}^L P \simeq \mathcal{P} \rtimes Gal_F$

${}^L P \to Gal_F$ and $\hat{P} = {}^L P \cap \hat{G}$