12/06/2018   Géo. Conjecture 2.

Div^1

$$\begin{tikzcd} E \arrow[r] \arrow[dr] & \mathbb{F}_q((\pi)) \\ & {[E:\mathbb{Q}_p] < \infty} \end{tikzcd} \qquad \mathbb{F}_q = \mathcal{O}_E/\pi .$$

$$\begin{tikzcd} \mathrm{Div}^1 = \mathrm{Spa}\, E^\diamond / \varphi^{\mathbb{Z}} \arrow[d] \\ \mathrm{Spa}\, \mathbb{F}_q \end{tikzcd}$$

moduli space of deg 1 effective Cartier div / curve.

$\mathrm{Div}^1(S) = \{ D \subseteq X_S \;$ deg 1 fiberwise $/ S \}$

$S \in \mathrm{Perf}_{\mathbb{F}_q}$

$\forall s \in S \quad \underbrace{D|_{X_{k(s),k(s)^+}}}_{\text{deg } 1} \subset X_{k(s),k(s)^+}$

Comparison of X_S and Div^1_S:

$$\begin{tikzcd} X_S^\diamond = (S \times \mathrm{Spa}(E)^\diamond)/\varphi_S^{\mathbb{Z}} \arrow[d] \\ \mathrm{Spa}(E)^\diamond \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Div}^1_S = (S \times \mathrm{Spa}(E)^\diamond)/\varphi_{E^\diamond}^{\mathbb{Z}} \arrow[d] \\ S \end{tikzcd}$$

$\varphi_S \circ \varphi_{E^\diamond} = \underbrace{\text{absolute Frob}}_{\text{acts trivially on } |\_| \text{ and on the étale site}}$ of $S \times \mathrm{Spa}(E)^\diamond$.

$$\Rightarrow \quad \begin{tikzcd} {|X_S| = |X_S^\diamond| = |\mathrm{Div}^1_S|} \arrow[d] \\ {|S|} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Div}^1_S \arrow[d] \\ S \end{tikzcd} \text{ proper cohom. sm.}$$

closed and open continuous map $|X_S| \to |S|$

→ gives support "X_S is fibered over S"

$$(X_S)^\sim_{\text{ét}} \cong (\mathrm{Div}^1_S)^\sim_{\text{ét}}$$

Local systems on $\mathrm{Div}^1_{\mathbb{F}_q}$:

$\Lambda \in \{ \overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell \}$

$\underline{\Lambda}(S) = \mathcal{C}^0(|S|, \Lambda)$ pro-étale sheaves

$\overline{\mathbb{Q}}_\ell = \varinjlim_{[\mathbb{Q}_\lambda : \mathbb{Q}_\ell] < \infty} \mathbb{Q}_\lambda$ inductive limit topology.

Prop: Sheaves of $\underline{\Lambda}$-modules loc. constant of finite rank on $(\mathrm{Div}^1_{\mathbb{F}_q})^\sim_{\text{pro-ét}} \cong \mathrm{Rep}_\Lambda(W_E)$

↳ not $\mathrm{Gal}(\overline{E}/E)$!

"$\pi_1(\mathrm{Div}^1_{\overline{\mathbb{F}}_q}) = W_E$".

Proof: $\mathrm{Div}^1_{\overline{\mathbb{F}}_q} = \big(\underbrace{\mathrm{Spa}(E)^\diamond \times_{\mathrm{Spa}\,\overline{\mathbb{F}}_q} \mathrm{Spa}(\overline{\mathbb{F}}_q)}_{\mathrm{Spa}(\breve{E})^\diamond} \big) / \varphi^{\mathbb{Z}}_{E^\diamond}$

$\breve{E} = \widehat{E^{un}} \circlearrowleft \sigma$

$\mathrm{Spa}\ \breve{E} \circlearrowleft \sigma = \varphi_{\overline{\mathbb{F}}_q}$

Local systems / $\mathrm{Div}^1_{\overline{\mathbb{F}}_q}$ = $\varphi_{E^\diamond}$ - eq. loc. systems on $(\mathrm{Spa}\ E)^\diamond \times (\mathrm{Spa}\ \overline{\mathbb{F}}_q)$

= σ - eq. loc. syst. on $(\mathrm{Spa}\,E)^\diamond$

= $\mathrm{Rep}_\Lambda(W_E)$. □

Later: G/E reductive group $\underbrace{\hat{G}}$ $\circlearrowleft W_E$

(Λ - points of
Langlands dual of G

$\longmapsto$ $\underline{\hat{G}}$ = sheaf of group on $(\mathrm{Div}^1_{\overline{\mathbb{F}}_q})^\sim_{\text{pro-ét}}$

$\mathrm{Aut}(\varphi) = S_\varphi$
$\downarrow$

$$\left\{ \begin{array}{c} \text{Langlands parameters} \\ \varphi : W_E \longrightarrow {}^L G \\ \quad\quad\quad\quad \nearrow \\ \hat{G} \rtimes W_E \end{array} \right\} \overset{\text{eq. of}}{\underset{\text{groupoids}}{\cong}} \left\{ \begin{array}{c} \underline{\hat{G}} \text{ - torsors} \\ \text{on } (\mathrm{Div}^1_{\overline{\mathbb{F}}_q})^\sim_{\text{pro-ét}} \end{array} \right\}$$

$\mathcal{E} = \underline{\hat{G}}$ - torsor on $\mathrm{Div}^1_{\overline{\mathbb{F}}_q}$ $\longmapsto$ $\mathrm{Aut}_{\mathcal{E}}$ = perverse sheaf on $\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$ ...

__The stack Bun_G:__

G/E reductive group.

$S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$

$\mathrm{Bun}_G(S)$ = groupoid of $\underline{G\text{-bundles}}$ on X_S

Tannakian sense: exact tensor functor $\mathrm{Rep}(G) \to \mathrm{Bun}_{X_S}$
$\nearrow$
(locally free sheaf of $\mathcal{O}_{X_S}$ - mod of finite rank.

Fact - Bun_G is a v-stack on $\mathrm{Perf}_{\mathbb{F}_q}$.

Proof - * Immediate reduction to $G = GL_n$

* T/S a v-cover $\quad X_T \times_{X_S} X_T = X_{T\times_S T}$ $\quad$ $$\begin{tikzcd} X_T \arrow[d] \\ X_S \end{tikzcd}$$

To prove: $\{ \text{vb on } X_S \} \xrightarrow{\sim} \{ \text{vb on } X_T + \text{descent dat. w.r.t. } X_{T\times_S T \times_S T} \rightrightarrows X_{T\times_S T} \rightrightarrows X_T \}$.

Two steps:

1) $C = \widehat{\overline{E}}$, $X_S \hat{\otimes}_E C$ is perfectoid

+ Scholze's descent of v.b. for the v-topo on Perf

(v.b. = stack for the v-topo. on Perf)

$\Rightarrow$ The result is true after applying $- \hat{\otimes}_E C$.

2) Descent from $X_S \hat{\otimes}_E C$ to X_S, use E direct factor in C:

$$C = E \oplus W \longleftarrow E\text{-Banach space}$$

$\square$.

Thm - Bun_G is a small v-stack on $\mathrm{Perf}_{\mathbb{F}_q}$ whose diagonal is represented by a locally spatial diamond.

$\longrightarrow$ $\exists \underbrace{Z}_{\substack{\text{perfectoid}\\ \text{space}}} \to \mathrm{Bun}_G$ $\underbrace{\text{surjective}}_{\text{v-topo. sens}}$

and $\forall \underbrace{S}_{\text{perfectoid}} \longrightarrow \mathrm{Bun}_G$, $\quad S \times_{\mathrm{Bun}_G} S$ = locally spatial diamond.

Rmk: loc. spatial diamond

= diamond D s.t. $|D|$ is loc. spatial

and D locally quasi-separated.

Proof - * Reduction to $G = GL_n$.

* S affinoid perfectoid, $\mathcal{E}$ v.b. $/X_S$ then for

$n \gg 0$ $\exists$ surjection $\mathcal{O}_{X_S}^m \twoheadrightarrow \mathcal{E}(n)$ $\quad$ (Kedlaya-Liu.

$T \in \mathrm{Perf}_S$

<u>Proposition</u> (Quot diamond). $\mathcal{F}$ v.b. on X_S.

$$\mathrm{Quot}_{\mathcal{F}} : T/S \longmapsto \{\text{loc. free quotients of } \mathcal{F}|_{X_T}\}$$

is represented by a locally spatial diamond $/S$.

<u>Proof</u>: $\mathrm{Quot}_{\mathcal{F}} = \coprod_{r \geq 1} \underbrace{\mathrm{Quot}^r_{\mathcal{F}}}$

~~locus~~ locus where the quotient has rank r

$\mathrm{Quot}^r_{\mathcal{F}} \hookrightarrow \mathrm{Quot}^1_{\Lambda^r \mathcal{F}}$ via Plucker embedding.

Then: reduce to $r = 1$.

If $\mathcal{F} \twoheadrightarrow \mathcal{L}$ = line bundle, then pro-étale locally $/S$ any $\mathcal{L} \cong \mathcal{O}(d)$ for some d.

The moduli of surjections $\mathcal{F} \twoheadrightarrow \mathcal{O}(d)$ for some d fixed, is an open subset of the BC-space of global sections of $\check{\mathcal{F}}(d)$ = locally spatial diamond. □

<u>Thm</u>. $\underbrace{\coprod_{\substack{m \geq 1 \\ n \in \mathbb{Z}}} \mathrm{Quot}_{\mathcal{O}(n)^m}}_{\text{diamond}} \xrightarrow{v\text{-surj.}} \mathrm{Bun}$ ← v-cover

$*$ For the diagonal: $\mathcal{E}_1, \mathcal{E}_2$ v.b. $/X_S$

$$\begin{tikzcd}
\underline{\mathrm{Isom}}(\mathcal{E}_1, \mathcal{E}_2) \arrow[rr, phantom, "\underset{\text{open}}{\subset}"] \arrow[dr] & & \underbrace{\underline{\mathrm{Hom}}(\mathcal{E}_1, \mathcal{E}_2)}_{BC(\check{\mathcal{E}}_1 \otimes \mathcal{E}_2) = \text{loc. spatial diamond}} \arrow[dl] \\
& S &
\end{tikzcd}$$

Used the following: $\mathcal{E}$ v.b. $/X_S$

$BC(\mathcal{E}) : T/S \longmapsto H^0(X_T, \mathcal{E}|_{X_T})$ is representable by a locally spatial diamond.

$\rightarrow$ locally on S: $0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}(d_1)^{m_1} \rightarrow \mathcal{O}(d_2)^{m_2}$

$\Rightarrow BC(\mathcal{E}) = \ker\left(BC(\mathcal{O}(d_1)^{m_1}) \longrightarrow BC(\mathcal{O}(d_2)^{m_2})\right)$

$BC(\mathcal{O}_{X_S}(d))$ = separated loc. spatial diamond $/S$.

Points of Bun_G: ~~[illegible]~~

From now : $/\overline{\mathbb{F}}_q$, $\mathrm{Bun}_G = \mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$

Consequence of the classification of G-bundles on X_C with $C/\overline{\mathbb{F}}_q$ alg. closed :

$$B(G) \xrightarrow{\sim} |\mathrm{Bun}_G|$$

↖ topo : open subsets $= \{ |U| \mid \underset{\text{open}}{U \subset} \mathrm{Bun}_G \}$.

Connected components :

Thm : $(E/\mathbb{Q}_p)$ - $\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma$ ← $\mathrm{Gal}(\overline{E}/E)$

is locally constant on Bun_G.

Proof : easy if G_{der} is simply connected since then, if $D = G/G_{der}$

$$\begin{tikzcd} B(G) \arrow[r] \arrow[rrr, bend right, "\kappa"'] & B(D) = \pi_1(D)_\Gamma & = & \pi_1(G)_\Gamma \end{tikzcd}$$

⇒ reduced to the case of tori

⇒ " " " " of $\mathbb{G}_m$ → $\mathcal{L}$ line bundle $/X_S$

$$\begin{cases} |S| \longrightarrow \mathbb{Z} \\ s \longmapsto \deg\left(\mathcal{L}|_{X_{k(s), k(s)^+}}\right) \end{cases}$$ is loc. constant.

If G_{der} is not simply connected, use :

$$\begin{tikzcd} \tau : (X_S)^{\sim}_{\acute{e}t} \arrow[r] \arrow[d, equal] & \widetilde{S}_{\acute{e}t} \\ (\mathrm{Div}^1_S)^{\sim}_{\acute{e}t} \arrow[ur] & \end{tikzcd}$$

$\forall n \geq 1$, $\mathrm{tr}\ R^2\tau_*\mu_n \xrightarrow{\sim} \underline{\mathbb{Z}/n\mathbb{Z}}$, even if $p \mid n$.

⇒ any coho. class in $H^2_{\acute{e}t}(X_S, \mu_n)$ is locally on S.

⟶ gives a new proof of a result of Rapoport - Richartz. □

Thus $\mathrm{Bun}_G = \coprod_{\alpha \in \pi_1(G)_\Gamma} \underbrace{\mathrm{Bun}_G^\alpha}_{\text{closed / open}}$ ← locus where $\kappa = -\alpha$

Thm. $\forall \alpha$ Bun_G^{α} is connected.

HN stratification: $\quad G$ quasi-split.

maximally split $\searrow$ $\underset{\substack{\uparrow \\ \text{max. torus}}}{A \subset T} \subset \underset{\substack{\uparrow \\ \text{Borel}}}{B} \subset G$

$$\mathrm{HN}: \left| \begin{array}{l} B(G) \longrightarrow X_*(A)^{\overset{\text{positive Weyl.}}{+}}_{\mathbb{Q}} \\ {[b]} \longmapsto \underset{\uparrow}{-w}.\,[\underset{\nwarrow \text{ Newton point}}{\nu_b}] \end{array} \right.$$

(longest lenght element in a Weyl group.

Thm. $\mathrm{HN}: |\mathrm{Bun}_G| \longrightarrow \underbrace{X_*(A)^+_{\mathbb{Q}}}$ is semi-continuous

usual order: $\left(\nu_1 \leq \nu_2 \leftrightarrow \nu_2 - \nu_1 \in \mathbb{Q}_{\geq 0} \cdot \check{\Phi}^+ \right.$

in the sense that $\forall \nu$ $\{\mathrm{HN} \geq \nu\}$ is open.

Proof: $\longrightarrow$ reduce to the case of GL_n, difficult th. by Kedlaya-Liu. $\square$

Ex: $\mathrm{Bun}_G^{ss} \xhookrightarrow{\text{open}} \mathrm{Bun}_G$

Recall: $\quad B(G) \xhookrightarrow{(\kappa,\nu)} \pi_1(G)_\Gamma \times X_*(A)^+_{\mathbb{Q}}$

$\kappa: B(G)_{\text{basic}} \xrightarrow{\sim} \pi_1(G)_\Gamma$.

$\forall \alpha \in \pi_1(G)_\Gamma$, $\nu \in X_*(A)^+_{\mathbb{Q}}$

$$|\mathrm{Bun}_G^{\alpha, \mathrm{HN}=\nu}| = \begin{cases} \emptyset \\ \text{one point} \end{cases}$$

$|\mathrm{Bun}_G^{\alpha, ss}| =$ one point

Conjecture: $\quad x, y \in |\mathrm{Bun}_G^{\alpha}|$, $\quad x \succ y \Leftrightarrow \mathrm{HN}(x) \geq \mathrm{HN}(y)$

$\longrightarrow$ proved for GL_n (Birkbeck, Feng, Hansen, Hong, Wang, Ye).

Structure of HN strata:

$b \in G(\breve{E})$, $\forall S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ $\quad \mathcal{E}_b / X_S$

$\underline{\mathrm{Aut}}(\mathcal{E}_b) = v$-sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$

$S \longmapsto \mathrm{Aut}(\mathcal{E}_b / X_S)$

Injection. $\underline{J_b(E)} \hookrightarrow \underline{Aut}(\mathcal{E}_b)$

This injection is split since

$$\begin{tikzcd} \varphi\text{-Mod}_{\breve{E}} \arrow[r] \arrow[rr, bend right, "Id"'] & Bun_{X_F} \arrow[r, "\overset{\text{semi-simplify HN-filt.}}{\downarrow}"] & \varphi\text{-Mod}_{\breve{E}} \end{tikzcd}$$

$$\underline{Aut}(\mathcal{E}_b) = \underbrace{\underline{Aut}(\mathcal{E}_b)^{\circ}}_{} \rtimes \underline{J_b(E)}$$

unipotent diamond / $Spa\, \overline{\mathbb{F}}_q$, successive extension of BC spaces of the form $BC(\mathcal{O}(\lambda))$ with $\lambda > 0$.

Ex) $$\underline{Aut}(\mathcal{O} \oplus \mathcal{O}(d)) = \begin{cases} \begin{pmatrix} \underline{E}^\times & BC(\mathcal{O}(d)) \\ 0 & \underline{E}^\times \end{pmatrix} & \text{if } d > 0 \\ \underline{GL_2(E)} & \text{if } d = 0 \end{cases}$$

b basic $\Rightarrow \underline{Aut}(\mathcal{E}_b) = \underline{J_b(E)}$

$$\dim \underline{Aut}(\mathcal{E}_b) = \langle \underset{\substack{\uparrow \\ \text{Newton point}}}{\nu_b}, 2\rho \rangle \xleftarrow{} \frac{1}{2}\Sigma \text{ positive roots.}$$

Thm. If $K(b) = -\alpha$, $\nu = -w.[\nu_b]$

$$\begin{cases} Bun_G^{\alpha, HN=\nu} \simeq \underbrace{\left[Spa(\overline{\mathbb{F}}_q) / \underline{Aut}(\mathcal{E}_b) \right]}_{v\text{-stack of } \underline{Aut}(\mathcal{E}_b)\text{-torsors}} \\ \mathcal{E} \longmapsto Isom(\mathcal{E}_b, \mathcal{E}) \end{cases}$$

(Ex. b basic $Bun_G^{\alpha, ss} = \left[Spa\, \overline{\mathbb{F}}_q / \underline{J_b(E)} \right]$

$\longrightarrow$ difficult : reduced to GL_n, $b = 1$ $\longrightarrow$ Kedlaya-Liu

$$\left\{ \begin{matrix} \text{vb on } X_S \text{ fiberwise on } S \\ \text{semi-stable slope } 0 \end{matrix} \right\} \underset{\substack{\uparrow \\ R\tau_*}}{\xrightarrow{\sim}} \left\{ \begin{matrix} \underline{E}\text{-proétale sheaves} \\ \text{loc. constant of finite} \\ \text{rk}/S \end{matrix} \right\}$$

$$\begin{cases} H = \text{locally profinite group} \\ S \text{ perfectoid space} \end{cases} \qquad \left.\begin{tikzcd} T \arrow[d] \\ S \end{tikzcd}\right) \ \underline{H}\text{ - torsor for the proétale topology.}$$

Using descent of separated étale morphisms for the v - topology :

$$\forall K \subset H \begin{pmatrix}\text{compact} \\ \text{open}\end{pmatrix} \qquad \begin{tikzcd} K\backslash T = \text{perfectoid} \arrow[d, "\text{étale}"] \\ S \end{tikzcd} \qquad T = \varprojlim_{K} K\backslash T$$