21/06/2018 **The Conjecture.** (1)

Recall: $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell\}$ $\qquad E \begin{cases} [E:\mathbb{Q}_p] \\ \mathbb{F}_q((\pi)) \end{cases}$

Work over $\mathrm{Spa}(\overline{\mathbb{F}}_q)$

$\forall \mathcal{F} \in D_{et}(\mathrm{Bun}_G, \Lambda) \quad \forall b \in G(\breve{E})$

$$j_b : [\cdot / \underline{\mathrm{Aut}}(\mathcal{E}_b)] \xrightarrow[\text{loc. closed}]{} \mathrm{Bun}_G$$

$\mathcal{H}^i(j_b^* \mathcal{F}) \in \mathrm{Rep}_\Lambda^{sm}(J_b(E)) \quad \forall i$

$$\underline{\mathrm{Aut}}(\mathcal{E}_b) = \underbrace{\underline{\mathrm{Aut}}(\mathcal{E}_b)^\circ}_{\text{acts trivially (connected)}} \rtimes \underline{J_b(E)}$$

$\mathrm{Perv}(\mathrm{Bun}_G, \Lambda) = \{ \mathcal{F} \in D_{et}(\mathrm{Bun}_G, \Lambda) \mid$. $\mathcal{F}$ reflexive $\overset{\text{thm}}{\Longleftrightarrow}$ $\forall b\ \forall i$ $\mathcal{H}^i(j_b^* \mathcal{F})$ admissible

. $\forall b$ $\mathcal{H}^i(j_b^* \mathcal{F}) = 0$ if $i > \langle \nu_b, 2\rho \rangle$

and $\mathcal{H}^i(j_b^! \mathcal{F}) = 0$ if $i < \langle \nu_b, 2\rho \rangle$ $\}$

Simple objects: $j_{b!*} \mathcal{F}_\pi [-\langle \nu_b, 2\rho\rangle]$

π = irred. of $J_b(E)$.

Hecke correspondences:

$$\begin{tikzcd} & \mathrm{Hecke}^{\leq \bar{\mu}} \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] \arrow[d] & \\ \mathrm{Bun}_G & \text{Moduli of} & \mathrm{Bun}_G \times \mathrm{Div}^1 \end{tikzcd}$$

$\left(\begin{array}{l} \mu \in X_*(T)^+ \\ \bar{\mu} \in \Gamma \backslash X_*(T)^+ \end{array} \right.$

Moduli of

$(\mathcal{E}_1, \mathcal{E}_2, D, u)$

Bun_G $\quad \mathrm{Div}^1$ $\quad$ modif. at D:

$S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ $\qquad \mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Bun}_G(S)$ $\qquad u : \mathcal{E}_1|_{X_S \setminus D} \xrightarrow{\sim} \mathcal{E}_2|_{X_S \setminus D}$

$\overleftarrow{h}(\_) = \mathcal{E}_2 \qquad \overrightarrow{h}(\_) = (\mathcal{E}_1, D)$.

Torsor:

$$\left.\begin{array}{c} \mathcal{T} \\ \downarrow \\ \mathrm{Bun}_G \times \mathrm{Div}^1 \ni (\mathcal{E}, D) \end{array}\right) L^+G/\varphi^{\mathbb{Z}} \text{ - étale torsor} \qquad \begin{array}{c} L^+G/\varphi^{\mathbb{Z}} \\ \downarrow \\ \mathrm{Spa}(E)^\diamond/\varphi^{\mathbb{Z}} = \mathrm{Div}^1 \end{array}$$

étale torsor of trivializations of $\mathcal{E}$ along the formal completion of the curve along D.

$(L^+G)(R, R^+) = G(B_{dR}^+(R^\#))$

$S = \mathrm{Spa}(R, R^+)$ $\quad D \subset X_S \quad$ $S \in \mathrm{Perf}_{\overline{\mathbb{F}_q}}$

$(X_S)^\wedge_D := \mathrm{Spf}(B_{dR}^+(R^\#))$

<u>Rmk</u>: $\mathrm{Spa}(\mathbb{Q}_p)^\diamond \times_{\mathrm{Spa}\,\mathbb{F}_p} S = \mathbb{D}_S^{*,1/p^\infty} / \mathbb{Z}_p^\times$

$\mathrm{Spa}(\mathbb{Q}_p)^\diamond = \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl},\flat}) / \mathbb{Z}_p^\times$

Then $\mathrm{Hecke}^{(\leq \bar\mu)} = \mathcal{T} \times_{L^+G/\varphi^{\mathbb{Z}}} \mathrm{Gr}^{(\leq \bar\mu)}/\varphi^{\mathbb{Z}}$ ← Beauville-Laszlo.

representable (in proper spatial diamonds) $\longrightarrow$ $\overrightarrow{h} \downarrow$

$\mathrm{Bun}_G \times \mathrm{Div}^1$

+ étale loc. trivial fibration in $\mathrm{Gr}^{(\leq \bar\mu)}/\varphi^{\mathbb{Z}}$.

$\rho \in \mathrm{Rep}_\Lambda({}^LG) \underset{\text{geo. Satake}}{\rightsquigarrow} IC_\rho$ $\quad L^+G/\varphi^{\mathbb{Z}}$-eq. perverse sheaf in $\mathrm{Gr}/\varphi^{\mathbb{Z}}$.

$\rightsquigarrow \underbrace{\mathcal{IC}_\rho}_{\substack{\cap \\ D_{et}(\mathrm{Hecke}, \Lambda)}} := \mathcal{T} \times_{L^+G/\varphi^{\mathbb{Z}}} IC_\rho$ on Hecke

(Perverse sheaf satisfies étale descent.)

$\rho \in \mathrm{Rep}_\Lambda({}^LG) \rightsquigarrow \mathrm{Hecke}_\rho : D_{et}(\mathrm{Bun}_G, \Lambda) \to D_{et}(\mathrm{Bun}_G \times \mathrm{Div}^1, \Lambda)$

$\mathcal{F} \mapsto \overrightarrow{h}_{\underset{(!)}{*}}(\overleftarrow{h}^* \mathcal{F} \otimes \mathcal{IC}_\rho)$

<u>Statement of the conjecture</u>:

$\varphi : W_E \longrightarrow {}^LG$ discrete (i.e. $S_\varphi / Z(\hat{G})^\Gamma$ finite)

$\Lambda = \overline{\mathbb{Q}_\ell}$ → should classify discrete series L-packets.

<u>Rmk</u>. * π smooth irred. rep. of $G(E)$ with coeff. in $\mathbb{C}$

$\tau \in \mathrm{Aut}(\mathbb{C})$ $\quad \pi$ discrete series $\Leftrightarrow {}^\tau\pi$ discrete series

↑ Casselman's criterion.

$\Rightarrow$ over $\overline{\mathbb{Q}_\ell}$ or $\mathbb{C}$, the notion is the same.

* False for tempered (unless the eigenvalues of Frob are Weil numbers).

(Dat introduced a notion of ℓ-adically tempered).

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* Say φ is cuspidal if $\begin{tikzcd} W_E \supset I_E \arrow[r, "\varphi|_{I_E}"] \arrow[rr, bend right, "1\text{-cocycle}"'] & {}^L G \arrow[r, two heads] & \widehat{G} \end{tikzcd}$ has finite image and φ is discrete.

$\longrightarrow$ Should classify supercuspidal L-packets.

* $\Lambda = \overline{\mathbb{Q}}_\ell$, $\mathcal{F}$ reflexive in Bun_G

$\mathcal{H}^i(j_b^* \mathcal{F}) \circlearrowleft J_b(E)$ + a $\overline{\mathbb{Z}}_\ell$-inv. lattice (up to commensurability).

($\rightsquigarrow$ mysterious)

<u>Conjecture</u>: Fix a Whittaker datum.

$\exists\, \mathcal{F}_\varphi \in \mathrm{Perv}(\mathrm{Bun}_G, \Lambda)$ + action of S_φ s.t.

$\boxed{\mathrm{Bun}_G = \coprod_{\alpha \in \pi_1(G)_\Gamma} \mathrm{Bun}_G^\alpha}$

1) $\forall \alpha \in \pi_1(G)_\Gamma$

$\mathcal{F}_\varphi|_{\mathrm{Bun}_G^\alpha} \circlearrowleft Z(\widehat{G})^\Gamma$ given by

$\alpha \in \pi_1(G)_\Gamma = X^*(Z(\widehat{G})^\Gamma)$

2) $(\Lambda = \overline{\mathbb{Q}}_\ell)$ $\forall b$ basic, $\pi = j_b^* \mathcal{F}_\varphi \circlearrowleft \underbrace{J_b(E)}_{\text{smooth + admissible}} \times S_\varphi$

and the action of S_φ restricted to $Z(\widehat{G})^\Gamma$ given by $-\kappa(b)$

+ $S_\varphi / Z(\widehat{G})^\Gamma$ = finite

$$\pi = \bigoplus_{\substack{\rho \in \mathrm{Rep}(S_\varphi) \\ \rho|_{Z(\widehat{G})^\Gamma} = -\kappa(b)}} \overset{J_b(E)}{\circlearrowright}\pi_\rho \otimes \rho \qquad \text{isotypic decompo.}$$

Then $(\underbrace{\pi_\rho}_{\text{irreducible}})_\rho$ = L-packet for a LL correspondence for J_b

such that for $b = 1$ ($J_b = G$.) π_1 = generic representation (choice of Whittaker datum.)

3) If moreover φ is cuspidal then $\mathcal{F}_\varphi$ is concentrated in Bun_G^{ss}

ie $j : \mathrm{Bun}_G^{ss} \hookrightarrow \mathrm{Bun}_G$ $\quad \mathcal{F}_\varphi = j_! j^* \mathcal{F}_\varphi$.

Rmk: Any $\mathcal{F} \in D_{\text{ét}}(\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}, \Lambda)$ has a Weil descent datum to $\mathbb{F}_q$:

$S \in \mathrm{Perf}_{\mathbb{F}_q}$ $\quad X_S$ functorial in S

$T \to S \quad X_T \to X_S$

$S \xrightarrow{\mathrm{Frob}} S \rightsquigarrow X_S \xrightarrow{\mathrm{Id}} X_S$

$B(G) = G(\breve{E})/_{\sigma\text{-conj.}} \qquad \forall b \in G(\breve{E}) \qquad b^\sigma = b^{-1} b (b^{-1})^{-\sigma}$

4) <u>Hecke property</u>: $\rho \in \mathrm{Rep}_\Lambda({}^LG)$ $\qquad W_E \xrightarrow{\varphi} {}^LG \xrightarrow{\rho} GL(V_\rho)$

$$\mu_\rho : \underbrace{\mathrm{Hecke}_\rho(\mathcal{F}_\varphi)}_{h_*(\overleftarrow{h}^* \otimes IC_\rho)} \xrightarrow{\sim} \mathcal{F}_\varphi \boxtimes \underbrace{\rho \circ \varphi}_{\text{local system of } \mathrm{Div}^1}$$

$\circlearrowleft S_\varphi \qquad \circlearrowleft S_\varphi$ (diagonal)

$(\mu_\rho)_\rho$ compatible with factorization sheaf property

<u>Particular case</u>: $\rho_1, \rho_2 \in \mathrm{Rep}({}^LG)$.

$$\mathrm{Hecke}_{\rho_1}(\mathcal{F}_\varphi) \xrightarrow[\mu_{\rho_1}]{\sim} \mathcal{F}_\varphi \boxtimes \rho_1 \circ \varphi$$

$$\begin{tikzcd}
(\mathrm{Hecke}_{\rho_2} \boxtimes \mathrm{Id})(\mathrm{Hecke}_{\rho_1}(\mathcal{F}_\varphi)) \arrow[r, "\sim"] \arrow[dr, "\nu"'] & \mathrm{Hecke}_{\rho_2}(\mathcal{F}_\varphi) \boxtimes \rho_1 \circ \varphi \arrow[d, "\wr\ \mu_{\rho_2}"] \\
& \mathcal{F}_\varphi \boxtimes \rho_2 \circ \varphi \boxtimes \rho_1 \otimes \varphi
\end{tikzcd}$$

$$\mathrm{Bun}_G \times [(\mathrm{Div}^1)^2 \setminus \Delta] \xhookrightarrow{j} \mathrm{Bun}_G \times (\mathrm{Div}^1)^2 \xhookleftarrow{i} \mathrm{Bun}_G \times \mathrm{Div}^1$$

$$i^* j_{!*} j^*(\nu) = \mu_{\rho_1 \otimes \rho_2} \xrightarrow[\text{geo. Satake}]{} i^* j_{!*} j^*[(\mathrm{Hecke}_{\rho_2} \boxtimes \mathrm{Id}) \circ \mathrm{Hecke}_{\rho_1}]$$
$$= \mathrm{Hecke}_{\rho_1 \otimes \rho_2}$$

5) <u>Local/global compatibility</u>:

$$\mathrm{Sh}_{K^p} \xrightarrow[\pi_{HT}]{} \mathcal{F}\ell^\mu \quad , \quad \overbrace{\mathcal{F}\ell^{\mu,\diamond}}^{\mathrm{Gr}^\mu} \xrightarrow{f} \mathrm{Bun}_G$$

Hodge-type ∞-level at p

(modifications of $\mathcal{E}_1$)

$(R\pi_{HT*}\underline{\Lambda})[\pi^p]$

$\updownarrow$

$f^* \mathcal{F}_\varphi$