20/06/2018 § . The BdR affine Grassmanian. ①

G/E $\quad (E \to [E:\mathbb{Q}_p]<\infty \quad \Gamma = \mathrm{Gal}(\overline{E}/E)$

$\searrow \mathbb{F}_q((\pi))$

Gr $\quad$ v-sheaf that is the $\diamond$ of the sheaf on Perf_E

$\downarrow$

$\mathrm{Spa}(E)^\diamond$

$\mathcal{F} \in \mathrm{Perf}_E, \quad \mathcal{F}^\diamond \in \mathrm{Perf}_{\mathbb{F}_q}$

$\mathcal{F}^\diamond(S) = \left\{ (S^\#, \iota, x) \;\middle|\; \begin{array}{l} S^\# \in \mathrm{Perf}_{E,b} \\ \iota: S \simeq S^{\#,b} \\ x \in \mathcal{F}(S^\#) \end{array} \right\}$

$\mathrm{Perf}_E \xrightarrow{\sim} \mathrm{Perf}_{\mathbb{F}_q} / \mathrm{Spa}(E)^\diamond$

$\mathcal{F} \longmapsto \mathcal{F}^\diamond / \mathrm{Spa}(E)^\diamond$

$\mathrm{Gr}: (R, R^+) \longmapsto \left\{ \begin{array}{l} G\text{-torsors on } \mathrm{Spec}\, \mathbb{B}_{dR}^+(R) \\ + \text{trivialization of its pullback to } \mathrm{Spec}\, \underbrace{\mathbb{B}_{dR}(R)}_{\mathbb{B}_{dR}^+(R)\left[\frac{1}{\xi}\right]} \end{array} \right\} / \sim$

$L^+G =$ group v-sheaf $\diamond$ of $(R, R^+) \longmapsto G(\mathbb{B}_{dR}^+(R))$.

$LG =$ idem with $G(\mathbb{B}_{dR}(R))$.

$LG / L^+G \xrightarrow{\sim} \mathrm{Gr}$ $\quad \to$ Any G-torsor on $\mathrm{Spec}(R)$ becomes trivial after replacing $\mathrm{Spa}(R, R^+)$ by an étale cover aff. perf.

$\uparrow$

étale quotient

<u>Beilinson - Drinfeld aff. grass.</u> :

Beauville - Laszlo $\quad \mathrm{Gr} / \varphi^{\mathbb{Z}}$

$\downarrow$

$\mathrm{Spa}(E)^\diamond / \varphi^{\mathbb{Z}} = \overbrace{\mathrm{Div}^1}^{\text{deg 1 effective Cartier divisors.}}$

moduli of $(\mathcal{E}, D, u)$

$\mathcal{E}$: G-bundle $/X_S$

D: $D \hookrightarrow X_S$

u: trivialization outside D

$u : \mathcal{E}_1|_{X_S \setminus D} \xrightarrow{\sim} \mathcal{E}|_{X_S \setminus D}$

$\uparrow$

meromorphic along D.

G quasi-split $\qquad T \subset B \subset G$

$\qquad\qquad\qquad$ max. torus $\quad$ Borel

Stratification indexed by ${}_\Gamma\backslash X_*(T)^+ \ni \bar\mu$

$\qquad\qquad\qquad\qquad\qquad\qquad (\mu \in X_*(T)^+)$

$\mathrm{Gr}^{\leq \bar\mu}$ = closed Schubert cell

$\quad$ = proper spatial diamond $/E^\diamond$

$$\begin{tikzcd} \mathrm{Gr}^{=\bar\mu} \arrow[r, hook, "\text{open}"'] \arrow[d, "\text{coho. smooth}"'] & \mathrm{Gr}^{\leq\bar\mu} \\ \mathrm{Spa}(E)^\diamond & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Gr}^{=\bar\mu} \arrow[d] \\ \mathcal{F}\ell^{\bar\mu} \end{tikzcd} \text{ iterated fibration in } \mathbb{A}^{1,\diamond}.$$

Ex) $E = \mathbb{F}_q((\pi)) \qquad (R, R^+)$ aff. perf. $/E$

$\quad E \ni \pi \longmapsto a \in R^{\circ\circ} \cap R^\times$

$\qquad\qquad\qquad \xi = \pi - a$ = deg 1 primitive

"In equal char., Fontaine's θ has a section $l: \pi \in \mathcal{O}_\pi$".

$$\mathbb{B}_{dR}^+ = R[\![X]\!] \xrightarrow{\theta} R$$

$$\begin{tikzcd} & X \arrow[rr, mapsto] & & 0 & \\ X+a & & E \arrow[ul] \arrow[ur] & & a \\ & \pi \arrow[ul, mapsto] & & \pi \arrow[ur, mapsto] & \end{tikzcd}$$

If $\mathrm{Gr}^{\ell R}/_{\mathrm{Spec}\,\mathbb{F}_q}$ twisted affine Grass. $\leftarrow$ parahoric subgroup

Gr = perfection of the adification of $\mathrm{Gr}^{\ell R} \otimes_{\mathbb{F}_q} \mathbb{F}_q((\pi))^\diamond$

$\quad$ = ind-perfectoid.

<u>The geometric Satake equivalence</u>:

G q. split, $E/\mathbb{Q}_p$, $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell\}$

Good theory of perverse sheaves on Gr:

* with q.c. support
* whose restriction to each $\mathrm{Gr}^{=\bar\mu}$, $\bar\mu \in {}_\Gamma\backslash X_*(T)^+$

is an étale local system of Λ-modules.

$$\begin{tikzcd} Gr^{=\bar\mu}_{\mathbb{F}_q} / \varphi^{\mathbb{Z}} \arrow[d] \\ \mathrm{Div}^1_{\mathbb{F}_q} / \varphi^{\mathbb{Z}} \end{tikzcd} \qquad \begin{tikzcd} Gr^{=\bar\mu} \otimes_{E^\diamond} \mathbb{C}_p \text{ s.c.} \arrow[d] \\ \mathcal{F}\ell \otimes \mathbb{C}_p \end{tikzcd}$$

<u>Main point</u>: $\quad j : Gr^{=\bar\mu} \hookrightarrow Gr^{\leq \bar\mu}$

$\mathcal{F}$ étale local system on $Gr^{=\bar\mu}$, $\mu' \in X_*(T)^+$

$(Rj_* \mathcal{F})|_{Gr^{=\bar\mu'}}$ = local system

→ clearly locally constant since L^+G-equivariant.

→ To be checked: finiteness of the stalks of this sheaf.

→ use Demazure resolution:

. replace Gr by $Gr \otimes_{E^\diamond} E'^\diamond$

. can suppose G split, $\mathcal{F} = \underline{\Lambda}$

$$\begin{tikzcd} \widetilde{Gr}^{=\mu} \arrow[r, hook, "\tilde{j}", "\text{open}"'] \arrow[d, "\simeq"'] & \widetilde{Gr}^{\leq\mu} \arrow[d, "f \text{ proper}"] & \\ Gr^{=\mu} \arrow[r, hook, "j", "\text{open}"'] & Gr^{\leq\mu} & \mathrm{Spa}(C) \arrow[l, "\bar x"'] \end{tikzcd}$$

= Demazure resolution

$$(Rj_* \underline{\Lambda})_{\bar x} = R\Gamma(f^{-1}(\bar x), R\tilde{j}_* \underline{\Lambda}).$$

$f^{-1}(\bar x) / \mathrm{Spa}(C)$ proper diamond

$G = GL_n$

$\mu = \mu_1 + \dots + \mu_d$ (minuscules)

$$\begin{tikzcd} \widetilde{Gr}^{\leq\mu} \arrow[d, "(G/P_{\mu_1})^\diamond"] \\ * \arrow[d] \\ \vdots \arrow[d, "(G/P_{\mu_d})^\diamond"] \\ \mathrm{Spa}(E)^\diamond \end{tikzcd}$$

Locally on $\widetilde{Gr}^{\leq\mu}$, these fibrations are split

→ cover $\widetilde{Gr}^{\leq\mu}$ by q.c. open U s.t.

$$U \subset (G/P_{\mu_1})^\diamond \times \dots \times (G/P_{\mu_d})^\diamond$$

$U = V^\diamond$

avec $V \subset G/P_{\mu_1} \times \dots \times G/P_{\mu_d}$

Check $U_c \cap f^{-1}(\bar{x}) = Z^\diamond$ $\quad Z \underset{\text{Zar. closed}}{\hookrightarrow} V_c$.

$u : V_c \setminus Z \hookrightarrow V_c$

Huber finiteness results : $\dim_\Lambda H^\bullet(Z, Ru_* \underline{\Lambda}) < \infty$

+ Čech cohomology. $\square$

* $\mathrm{Sat}(Gr_{\overline{\mathbb{F}}_q}/\varphi^{\mathbb{Z}}, \Lambda)$ = φ-eq. perverse sheaves on $Gr_{\overline{\mathbb{F}}_q}$

Thm. $\mathrm{Sat}(Gr_{\overline{\mathbb{F}}_q}/\varphi^{\mathbb{Z}}, \Lambda) \simeq \mathrm{Rep}_\Lambda({}^L G) \leftarrow \hat{G} \rtimes W_E$

$$\begin{tikzcd} \mathrm{Sat}(Gr_{\overline{\mathbb{F}}_q}/\varphi^{\mathbb{Z}}, \Lambda) \arrow[r, "\simeq"] & \mathrm{Rep}_\Lambda({}^L G) \\ \Lambda\text{-loc. syst.}/\mathrm{Div}^1_{\overline{\mathbb{F}}_q} \arrow[u, "\text{pullback}"] \arrow[r, "\simeq"] & \mathrm{Rep}_\Lambda(W_E) \arrow[u] \end{tikzcd}$$

→ Uses $*$ product defined via fusion on $\mathrm{Sat}(Gr, \Lambda)$

$* \leftrightarrow \otimes$

I finite set $\quad Gr_I \to (\mathrm{Spa}(E)^\diamond)^I$

$Gr_I/(\varphi^I)^{\mathbb{Z}}$ = moduli of $(\mathcal{E}, (D_i)_{i\in I}, u)$, $\quad D_i \subset X_S$

$\downarrow$ $\quad \mathcal{E}$: G-bundle$/X_S$

$(\mathrm{Div}^1)^I$

$\cancel{u_i : \mathcal{E}_1|_{X_S \setminus D_i} \xrightarrow{\sim} \mathcal{E}|_{X_S\setminus D_i}}$

$u : \mathcal{E}_1|_{X_S \setminus \cup D_i} \xrightarrow{\sim} \mathcal{E}|_{X_S \setminus \cup D_i}$

factorization B_{dR}-affine Grass.

→ fusion diagram:

$$\begin{tikzcd} Gr \arrow[r, hook, "i"] \arrow[d] & Gr_{\{1,2\}} \arrow[d] \\ \mathrm{Spa}(E)^\diamond \arrow[r, hook] & \mathrm{Spa}(E)^\diamond \times \mathrm{Spa}(E)^\diamond \end{tikzcd}$$

$$\begin{tikzcd} Gr \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & Gr_{\{1,2\}} \arrow[d] \arrow[dr, phantom, "\square"] & (Gr \times Gr)|_{\mathrm{Spa}(E)^\diamond \setminus \Delta} \arrow[l, hook', "j"'] \arrow[d] \\ \mathrm{Spa}(E)^\diamond \arrow[r, hook, "\Delta"] & \mathrm{Spa}(E)^\diamond \times \mathrm{Spa}(E)^\diamond & (\mathrm{Spa}(E)^\diamond)^2 \setminus \Delta \arrow[l, hook'] \end{tikzcd}$$

Perverse sheaves on Bun_G:

Sheaves on small v-stacks:

$\mathcal{X}$ = v-stack on $\mathrm{Perf}_{\mathbb{F}_q}$ (ie stack on $\mathrm{Perf}_{\mathbb{F}_q}$ with v-topo).

What is a sheaf on $\mathcal{X}$?

Sheaf = rule : $\forall \underset{\underset{\mathrm{Perf}}{\cap}}{S} \xrightarrow{f} \mathcal{X} \longmapsto$ v-sheaf $\mathcal{F}_S$ on S

$$+ \text{ if } \begin{tikzcd} S \arrow[r] \arrow[dr, "a"'] & S' \arrow[d, "b"] \\ & \mathcal{X} \end{tikzcd} \qquad \begin{array}{l} a \in \mathrm{Ob}(\mathcal{X}(S)) \\ b \in \mathrm{Ob}(\mathcal{X}(S')) \end{array}$$

+ iso $u : f^* b \xrightarrow{\sim} a$

An iso. $\alpha_{f,u} : f^* \mathcal{F}_{S'} \xrightarrow{\sim} \mathcal{F}_S$ satisfying a cocycle condition.

- Cartesian sheaves
- Pb: not a set in general $\to$ smallness hypothesis

$\mathcal{X}$ small $\quad \exists \underset{\underset{\text{perfectoid}}{\cup}}{U} \to \mathcal{X}$ v-surjective and

$\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ representable in loc. spatial diamonds.

$U_n = U \underset{\mathcal{X}}{\times} \cdots \underset{\mathcal{X}}{\times} U \to \underbrace{U_\bullet}_{\text{simplicial diamond}}$

v-Sheaves$/_{\mathcal{X}} \xrightarrow{\sim}$ Cartesian v-sheaves $/_{U_\bullet}$

Etale sheaves:

Pb: we need to work v-locally and mix this with the etale topo.

Scholze's solution: Λ finite ring $\quad (|\Lambda|, p) = 1$

$\mathcal{X}$ = small v-stack

Def: $D_{et}(\mathcal{X}, \Lambda) = \{ \mathcal{F} \in D(\mathcal{X}, \Lambda) \mid \forall S \xrightarrow{f} \mathcal{X}$ with S strictly totally disconnected $f^* \mathcal{F} \in D(S_{et}, \Lambda) \}$

⚠ Not $D(\mathcal{X}_{et}, \Lambda)$

S strictly disconnected $\Rightarrow D(S_{et}, \Lambda) = D(|S|, \Lambda)$.

$\longrightarrow$ X = locally spatial diamond $D_{et}(X, \Lambda) = D(X_{et}, \Lambda)$

Morphisms: $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ representable in locally spatial diamonds.

- $D_{et}(\mathcal{X}, \Lambda) \underset{f^*}{\overset{f_*}{\rightleftarrows}} D_{et}(\mathcal{Y}, \Lambda)$

  f^* easy to define

$\exists$ $\mathcal{D}_{et}(\mathcal{X}, \Lambda)$ ∞-stable category s.t. $\pi_0 \mathcal{D}_{et}(\mathcal{X}, \Lambda) = D_{et}(\mathcal{X}, \Lambda)$.

- $f : \mathcal{X} \to \mathcal{Y}$ representable in loc. spatial diamonds $+ \underbrace{\text{partially proper}}$.

  $\hookrightarrow$ "overconvergent morphism"

$\rightarrow$ $\left(\mathrm{Bun}_G / \mathrm{Spa}(\mathbb{F}_q)\right)$ is partially proper $\mathrm{Bun}_G(R, R^+)$
$= \mathrm{Bun}_G(R, R^\circ)$.

$\rightarrow$ X/K separated rigid space

$X/_{\mathrm{Spa}(K)}$ is p.p. $\Leftrightarrow$ $\forall\, U \underset{\text{q.c. open}}{\subseteq} X$

$\overline{U}^X$ = proper pseudo-adic space $/\mathrm{Spa}(K)$.

$f : \mathcal{X} \to \mathcal{Y}$ $\longrightarrow$ $Rf_! : D_{et}(\mathcal{X}, \Lambda) \to D_{et}(\mathcal{Y}, \Lambda)$

$f : X \to Y$, pp. (X, Y loc. spatial diamonds)

$Rf_!$ = derived functor of $\Gamma_c(X/Y, \mathcal{F})$
$:= \{ s \in \mathcal{F}(X) \mid \mathrm{Supp}(s) \xrightarrow[\text{proper}]{} Y \}$

$\longrightarrow$ extends to $f_! : D_{et}(\mathcal{X}, \Lambda) \longrightarrow D_{et}(\mathcal{Y}, \Lambda)$ thanks to proper base change.

Proposition. If sup $\deg \operatorname{tr}$ (géo fibers of f) $< +\infty$, ($f : \mathcal{X} \to \mathcal{Y}$) $f_!$ has a right adjoint $f^! : D_{et}(\mathcal{Y}, \Lambda) \to D_{et}(\mathcal{X}, \Lambda)$.

$\longrightarrow$ Lurie's adjunction.

Remark: Deligne in SGA 4: uses Godement resolutions.

Coho. smooth morphisms: $\mathcal{X} \xrightarrow{f} \mathcal{Y}$

Def: f coho. smooth if $\exists\, \mathbb{L} \in D_{et}(\mathcal{X}, \Lambda)$ invertible s.t. $f^! = f^* \otimes \mathbb{L}$. ($\to$ v.-loc $\Lambda[n]$, $n \in \mathbb{Z}$)

Ex: $$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & \operatorname{Spa} \mathbb{Q}_p & \end{array}$$ q.s. rigid spaces

f smooth $\Rightarrow$ $f^\lozenge$ coho. smooth

Huber Poincaré duality.

Property. Coho. smoothness is v.-local / base.

Ex: $S \in \operatorname{Perf}_{\mathbb{F}_q}$, $\mathcal{E}$ = v.b. $/X_S$, fiberwise $/S$ $\mathcal{E}$ has >0 slopes.

$$BC(\mathcal{E}) : T/S \longmapsto H^0(X_T, \mathcal{E}|_{X_T})$$

$$\begin{array}{c} BC(\mathcal{E}) \\ \downarrow \text{coho. smooth} \\ S \end{array}$$

$\to$ prove that v.-locally $/S$, $\mathcal{E}$ has a filtration by v.b., graded pieces $\simeq \mathcal{O}\left(\frac{d}{n}\right)$, $n \geq 1$.

$$BC\left(\mathcal{O}\left(\tfrac{1}{n}\right)\right) \simeq \mathbb{B}_S^{n, 1/p^\infty}.$$

v.-locally $/S$

$$\text{coho. smooth} \left(\begin{array}{c} BC(\mathcal{E}) \\ \downarrow \\ \times \\ \downarrow \\ \vdots \\ \downarrow \\ S \end{array} \right) BC\left(\mathcal{O}\left(\tfrac{1}{n}\right)\right)\text{-torsor}$$

Last ex:

X = loc. spatial diamond $/S$

K = pro-p-group

$$\begin{tikzcd} X \arrow[d, "f"] & \circlearrowleft \underline{K} \text{ free action } /S \\ S \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d, "a"'] \\ X/\underline{K} \arrow[d, "b"'] \\ S \end{tikzcd}$$

$X/\underline{K}$] loc. spatial

$X/_S$ coho smooth

$\Rightarrow \left(X/\underline{K}\right)/S$ coho. smooth

$\underline{\text{Ex}}$: $\mathcal{E}$ v.b. $/X_S$ <0 slopes fiberwise$/_S$.

$\underbrace{R^1\tau_*}\mathcal{E}$ = sheaf associated to $T/_S \longmapsto H^1(X_T, \mathcal{E}|_{X_T})$.

coho sm $/S$

$\longrightarrow R^1\tau_*\mathcal{O}(-1) = \mathbb{G}_a^\diamond/\underline{E} = \left(\mathbb{G}_a^\diamond/\mathcal{O}_{\underline{E}}\right)/\left(\underline{E/\mathcal{O}_E}\right)$

coho sm.

Some coho. smooth charts on Bun_G:

P parabolic subgroup of G

$\mathfrak{p} = \mathrm{Lie}\, P$, $\mathfrak{g} = \mathrm{Lie}\, G$

$$\begin{tikzcd} \mathrm{Bun}_P^{>0} \arrow[r, phantom, "\underset{\text{open}}{\subseteq}"] \arrow[d] & \mathrm{Bun}_P \arrow[r, "\mathcal{E} \,\mapsto\, \mathcal{E}\times_P G"] \arrow[d] & \mathrm{Bun}_G \\ \mathrm{Bun}^{>0}_{GL(\mathfrak{g}/\mathfrak{p})} \arrow[r, phantom, "\underset{\text{open}}{\subseteq}"] & \mathrm{Bun}_{GL(\mathfrak{g}/\mathfrak{p})} & \text{via } P \xrightarrow{Ad} GL(\mathfrak{g}/\mathfrak{p}) \arrow[l] \end{tikzcd}$$

HN slopes >0

$\mathrm{Bun}_P^{>0} = \{ \mathcal{E} = P\text{-bundle s.t. } \mathcal{E} \times_P \mathfrak{g}/_{\mathfrak{p}}$ has >0 slopes.

$\underline{\text{Ex}}$) $G = GL_n$, $P = B$

"anti HN"

$0 = \mathcal{E}_0 \subsetneq \dots \subsetneq \mathcal{E}_n = \mathcal{E}$ s.t. $\deg \mathcal{E}_i/_{\mathcal{E}_{i-1}} < \deg \mathcal{E}_{i+1}/_{\mathcal{E}_i}$

Thm $\mathrm{Bun}_P^{>0} \longrightarrow \mathrm{Bun}_G$ is coho. smooth.

Application. Fix $b \in G(\breve{E})$ s.t. ν_b defined $/E$

$$\nu_b : \mathbb{D} \longrightarrow G.$$

$P = P_{\nu_b}$, $M = G(\nu_b)$ Levi of P.

$b = b_M \in M(\breve{E})$

Def: $\mathcal{M}_b$ = moduli of P-bundles $\mathcal{E}$ s.t. geometrically fiberwise $\mathcal{E} \times_P M \simeq \mathcal{E}_{b_M}$. (as an M-bundle).

$$\pi_b : \mathcal{M}_b \longrightarrow \mathrm{Bun}_G$$

Thm: $\pi_b : \overbrace{\mathcal{M}_b}^{\text{connected}} \longrightarrow \mathrm{Bun}_G$ is coho. smooth.

$\Rightarrow$ π_b is open.

$$\underset{\substack{\cap \\ \mathrm{Im}(\pi_b)}}{[b]} \in B(G) = |\mathrm{Bun}_G|$$

$\Rightarrow$ Any generalization of $[b]$ lies in $\mathrm{Im}(\pi_b)$.

$\Rightarrow$ $\mathrm{Bun}_G^{K(b)}$ is connected.

Ex: $G = GL_2$ Moduli of extensions $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0$

$E = \mathbb{Q}_p$ $\quad R^1\pi_*\mathcal{O}(-1) = \mathbb{G}_a^\diamond / \underline{\mathbb{Q}_p}$

$\mathcal{M}_b$ = moduli of ext. $0 \to \underset{\substack{| \\ \deg 0}}{\mathcal{L}} \to \mathcal{E} \to \underset{\substack{| \\ \deg 1}}{\mathcal{L}'} \to 0$

$$\mathcal{M}_b = \left[\mathbb{G}_a^\diamond \Big/ \begin{pmatrix} \underline{\mathbb{Q}_p}^\times & \underline{\mathbb{Q}_p} \\ 0 & \underline{\mathbb{Q}_p}^\times \end{pmatrix} \right] \supseteq \left[\Omega^\diamond \Big/ \begin{pmatrix} \underline{\mathbb{Q}_p}^\times & \underline{\mathbb{Q}_p} \\ 0 & \underline{\mathbb{Q}_p}^\times \end{pmatrix} \right]$$

$(0) \leftarrow$ locus where $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{O}(1)$

locus where $\mathcal{E} \simeq \mathcal{O}\left(\frac{1}{2}\right)$

$\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$ $K = \underline{\Lambda}[0]$ dualizing complex on Bun_G $\leftrightarrow$ $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$

$$\mathbb{D}\mathcal{F} = R\mathcal{H}om(\mathcal{F}, K)$$

Def - $\mathcal{F} \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ is reflexive if $\mathcal{F} \xrightarrow{\sim} \mathbb{D}\mathbb{D}\mathcal{F}$

$\forall b \in G(\breve{E})$

$$j_b : \left[\cdot / \underline{\mathrm{Aut}}(\mathcal{E}_b) \right] \xrightarrow[\text{loc. closed}]{} \mathrm{Bun}_G$$

$$\underline{\mathrm{Aut}}(\mathcal{E}_b) = \underline{\mathrm{Aut}}(\mathcal{E}_b)^\circ \rtimes \underline{J_b(E)}$$

Thm (Scholze) - $\mathcal{F} \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ reflexive.

$\Leftrightarrow$ $\forall b$ $\forall i \in \mathbb{Z}$ $\mathcal{H}^i(j_b^* \mathcal{F})$ as a smooth repr. of $J_b(E)$ is admissible

(sheaf of Λ-mod on $\left[\cdot / \underline{\mathrm{Aut}}(\mathcal{E}_b) \right]$.

Corollary - $\forall b$ π smooth admissible rep. of $J_b(E)$

$j_{b!} \underbrace{\mathcal{F}_\pi}$ and $R j_{b*} \mathcal{F}_\pi$ are reflexive.

$\hookrightarrow$ local system $\left[\cdot / \underline{\mathrm{Aut}}(\mathcal{E}_b) \right]$

Proof: satisfies the hypothesis of the thm.

+ $\mathcal{F}$ reflexive $\Rightarrow$ $\mathbb{D}\mathcal{F}$ reflexive

Cor - $\mathrm{Perv}(\mathrm{Bun}_G, \Lambda) = \{ \mathcal{F} \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \mid$ $\mathcal{F}$ reflexive and $\forall b$ $\mathcal{H}^i(j_b^* \mathcal{F}) = 0$ if $i > \langle \nu_b, 2\rho \rangle$, $\mathcal{H}^i(j_b^! \mathcal{F}) = 0$ if $i < \langle \nu_b, 2\rho \rangle \}$

$(j_b^! = \mathbb{D} j_b^* \mathbb{D})$.

is an abelian category.

Simple object : $\underbrace{j_{b!*}}_{\mathrm{Im}({}^p j_! \to {}^p R j_*)} \mathcal{F}_\pi$ π = smooth irred of $J_b(E)$