

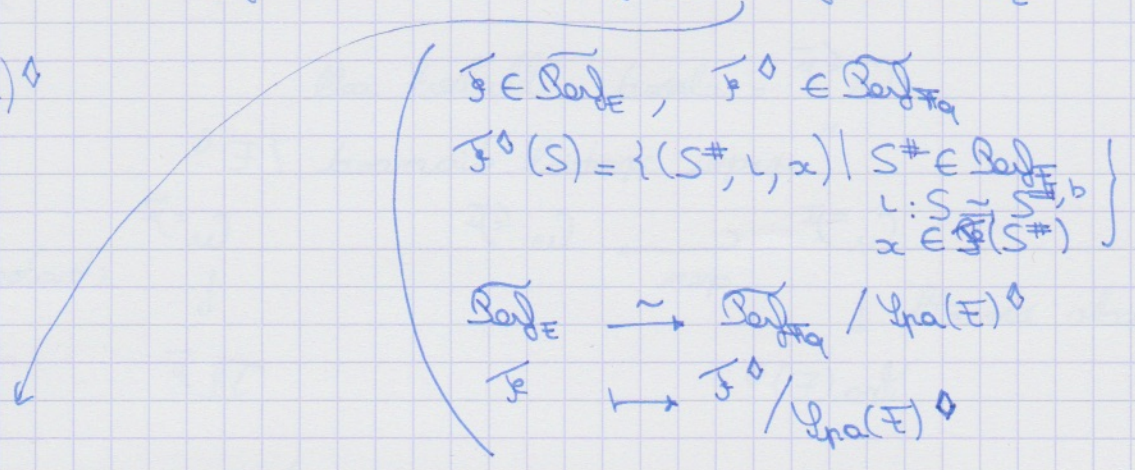
26/06/2018

§. The Bdr affine Grassmannian.

(1)

$$G/E \quad (E \rightarrow \{E: \text{Rk} \leq \infty\} \rightarrow \mathbb{F}_q((T))) \quad \Gamma = \text{Gal}(\bar{E}/E)$$

$G_n$  is a sheaf that is the  $\diamond$  of the sheaf on  $\text{Perf}_E$   
 $\downarrow$   
 $\text{Ypa}(E)^\diamond$



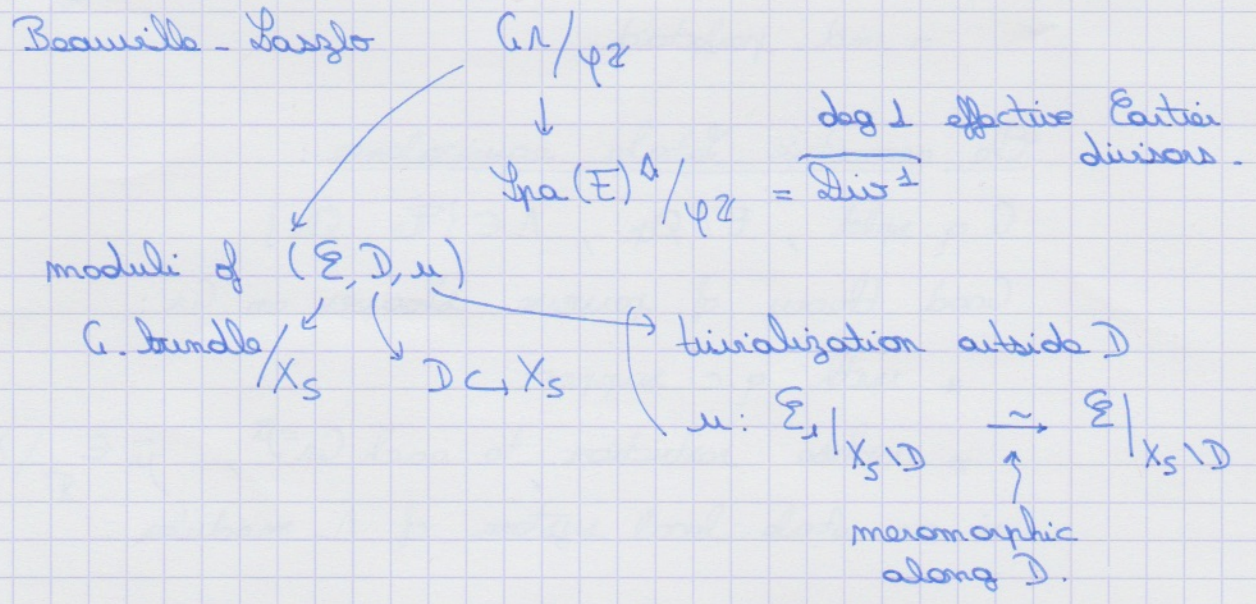
$G_n: (R, R^+) \mapsto \left\{ \begin{array}{l} G\text{-torsors on } \text{Spec } \overline{\text{Bdr}}(R) \\ + \text{trivialization of its pullback to } \text{Spec } \text{Bdr}(R) \end{array} \right\} \sim \overline{\text{Bdr}}(R) \left[ \frac{1}{T} \right]$

$L^+G = \text{group } \nu\text{-sheaf } \diamond \text{ of } (R, R^+) \mapsto G(\overline{\text{Bdr}}(R))$

$LG = \text{idem with } G(\text{Bdr}(R))$

$LG/L^+G \xrightarrow{\sim} G_n \rightarrow \text{any } G\text{-torsor on } \text{Spec}(R) \text{ becomes trivial after replacing } \text{Ypa}(R, R^+) \text{ by an etale cover aff. perf.}$

Beilinson - Drinfeld aff. grass.:





$G$  quasi-split  $T \subset B \subset G$   
 "max. torus" "Borel"

Stratification indexed by  $\Gamma \backslash X_*(T)^+ \ni \bar{\mu}$   
 $(\mu \in X_*(T)^+)$

$G_n \subseteq \bar{\mu}$  = closed Schubert cell

"proper spatial diamond /  $E^\diamond$

$G_n = \bar{\mu} \xrightarrow{\text{open}} G_n \subseteq \bar{\mu}$

$G_n = \bar{\mu}$

iterated fibration

coho. smooth  $\downarrow$

$\mathcal{Y}_{\text{pa}}(E)^\diamond$

$\mathcal{Y}_{\text{el}} \bar{\mu}$

in  $A^1, \diamond$

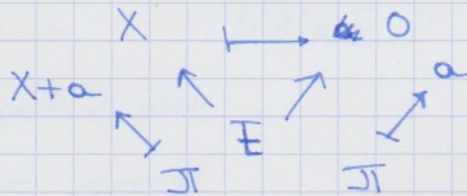
$\Sigma_2$   $E = \mathbb{F}_q((\mathcal{J}))$   $(R, R^+)$  aff. perf. /  $E$

$E \ni \mathcal{J} \mapsto a \in R^\infty \cap R^\times$

$\xi = \mathcal{J} \cdot a = \text{deg } 1$  primitive

"In equal char., Fontaine's  $\Theta$  has a section  $\mathcal{L}(\mathcal{J}) \in \mathcal{O}_{\mathcal{J}}$ ".

$\mathbb{R}_{\text{DR}}^+ = R \mathcal{J} \times \mathcal{J} \xrightarrow{\Theta} R$



$\mathcal{Y}_{\text{el}} G_n^{\text{DR}} / \mathcal{Y}_{\text{loc}} \mathbb{F}_q$  twisted affine Grass.  $\leftarrow$  parabolic subgroup

$G_n$  = perfection of the adification of  $G_n^{\text{DR}} \otimes_{\mathbb{F}_q} \mathbb{F}_q((\mathcal{J}))^\sharp$   
 = unid. perfectoid.

The geometric Satake equivalence:

$G$  q-split,  $E/\mathcal{O}_n$ ,  $\lambda \in \{\overline{\mathbb{F}_e}, \overline{\mathbb{Q}_e}\}$

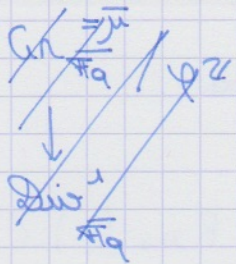
Good theory of perverse sheaves on  $G_n$ :

\* with q.c. support

\* whose restriction to each  $G_n = \bar{\mu}$ ,  $\bar{\mu} \in \Gamma \backslash X_*(T)^+$

is an étale local system of  $\lambda$ -modules.





$$G_n = \bar{\mu} \otimes_{\mathbb{F}} \mathbb{C}_n \text{ s.c.}$$

(2)

$$\mathbb{F} \otimes \mathbb{C}_n$$

Main point:  $f: G_n = \bar{\mu} \hookrightarrow G_n \leq \bar{\mu}$

$\mathbb{F}$  stable local system on  $G_n = \bar{\mu}$ ,  $\mu' \in X_*(\Gamma)^+$

$$(Rf_* \mathbb{F})|_{G_n = \bar{\mu}} = \text{local system}$$

→ clearly locally constant since  $L^+G$ -equivariant.

→ To be checked: finiteness of the stalks of this sheaf.

→ use Demazure resolution =

replace  $G_n$  by  $G_n \otimes_{\mathbb{F}} \mathbb{F}'^\diamond$

can suppose  $G$  split,  $\mathbb{F} = \underline{\Lambda}$

$$\begin{array}{ccc} \widetilde{G}_n = \bar{\mu} & \xrightarrow[\text{open}]{\tilde{f}} & \widetilde{G}_n \leq \bar{\mu} \\ \downarrow & & \downarrow \text{proper} \end{array} = \text{Demazure resolution}$$

$$\begin{array}{ccc} G_n = \bar{\mu} & \xrightarrow[\text{open}]{f} & G_n \leq \bar{\mu} \\ & & \xleftarrow{\bar{x}} \text{Spa}(C) \end{array}$$

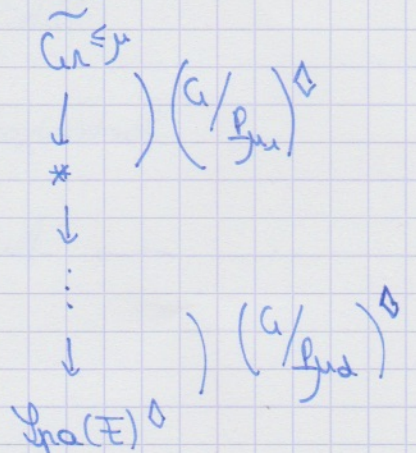
$$(Rf_* \underline{\Lambda})_{\bar{x}} = RT^1(f^{-1}(\bar{x}), R\tilde{f}_* \underline{\Lambda}).$$

$f^{-1}(\bar{x}) / \text{Spa}(C)$  proper diamond

$$G = G_1 \times \dots \times G_d$$

$$\mu = \mu_1 + \dots + \mu_d$$

(minuscules)



Locally on  $\widetilde{G}_n \leq \bar{\mu}$ , these fibrations are split

→ cover  $\widetilde{G}_n \leq \bar{\mu}$  by q.c. open  $U$  s.t.

$$U \subset (G/P_{\mu_1})^\diamond \times \dots \times (G/P_{\mu_d})^\diamond$$

avec  $V \subset G/P_{\mu_1} \times \dots \times G/P_{\mu_d}$



Check  $U_c \cap f^{-1}(\bar{x}) = Z^\diamond \quad Z \hookrightarrow V_c$   
 Zar. closed.

$$u: V_c \setminus Z \hookrightarrow V_c$$

Hilbert finiteness results:  $\dim_{\mathbb{A}} H^*(Z, R_{u_*} \underline{\Lambda}) < \infty$   
 + Čech cohomology  $\square$

\*  $\text{Sew}(G_n/\mathbb{F}_q / \varphi_Z, \Lambda) = \varphi$  - eq. reverse sheaves on  $G_n/\mathbb{F}_q$

Thm -  $\text{Sew}(G_n/\mathbb{F}_q / \varphi_Z, \Lambda) \simeq \text{Pop}_{\Lambda}(G)$   
 pullback  $\uparrow$   $\uparrow$   $\leftarrow \widehat{G} \times W_E$   
 $\Lambda$ -loc. syst. /  $\text{Dis}_{\mathbb{F}_q}^1 \simeq \text{Pop}_{\Lambda}(W_E)$

$\rightarrow$  Uses \* product defined via fusion on  $\text{Sew}(G_n, \Lambda)$   
 $* \leftrightarrow \otimes$

I finite set  $G_n$   
 $\downarrow$   
 $(\text{Spa}(E)^\diamond)^I$

$G_n / (\varphi^n)_Z = \text{moduli of } (\Sigma, (D_i)_{i \in I}, (W_i)_{i \in I})$   
 $\downarrow$   $\downarrow$   $\downarrow$   
 $(\text{Dis}^1)^I$   $G$ -bundle /  $X_S$   $\mu$

factorization Bdr. affine Grass.

$$\mu: \Sigma_i / X_S \setminus D_i \xrightarrow{\sim} \Sigma / X_S \setminus D_i$$

$$\mu: \Sigma_{i, X_S \setminus U_i} \xrightarrow{\sim} \Sigma / X_S \setminus U_i$$

$\rightarrow$  fusion diagram:

$$\begin{array}{ccc} G_n & \hookrightarrow & G_n \times \{1, 1\} \\ \downarrow & \searrow & \downarrow \\ \text{Spa}(E)^\diamond & \hookrightarrow & \text{Spa}(E)^\diamond \times \text{Spa}(E)^\diamond \end{array}$$

$$\begin{array}{ccc} G_n & \xrightarrow{i} & G_n \times \{1, 1\} & \xrightarrow{j} & (G_n \times G_n) / \text{Spa}(E)^\diamond \setminus \Delta \\ \downarrow & \square & \downarrow & \square & \downarrow \end{array}$$

$$\text{Spa}(E)^\diamond \hookrightarrow \text{Spa}(E)^\diamond \times \text{Spa}(E)^\diamond \longleftarrow (\text{Spa}(E)^\diamond)^\square \setminus \Delta$$



Reverse sheaves on  $\text{Bun}_G$ .

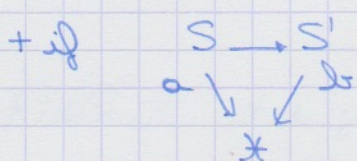
(3)

Sheaves on small  $v$ -stacks:

$\mathcal{X} = v$ -stack on  $\text{Perf}_{\mathbb{F}_q}$  (ie stack on  $\text{Perf}_{\mathbb{F}_q}$  with  $v$ -topo).

What is a sheaf on  $\mathcal{X}$ ?

Sheaf = rule:  $\forall S \xrightarrow{f} \mathcal{X} \mapsto v$ -sheaf  $\mathcal{F}_S$  on  $S$   
 $\bigcap_{\text{Perf}}$



$a \in \text{Ob}(\mathcal{X}(S))$

$b \in \text{Ob}(\mathcal{X}(S'))$

+ iso  $\mu: f^*b \xrightarrow{\sim} a$

An iso.  $\alpha_{f,\mu}: f^*\mathcal{F}_{S'} \xrightarrow{\sim} \mathcal{F}_S$  satisfying a cocycle condition.

Cartesian sheaves

Ob: not a set in general  $\rightarrow$  smallness hypothesis

$\mathcal{X}$  small  $\exists U \rightarrow \mathcal{X}$   $v$ -surjective and  
 $\bigcup$   
perfectoid

$\Delta_{\mathcal{X}}: \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  representable in loc. spatial diamonds.

$U_n = U \times_{\mathcal{X}} \dots \times_{\mathcal{X}} U \xrightarrow{\sim} U$   
simplicial diamond

$v$ -Sheaves/ $\mathcal{X} \xrightarrow{\sim}$  Cartesian  $v$ -sheaves/ $U$ .

Etale sheaves:

Ob: we need to work  $v$ -locally and mix this with the etale topo.

Scholze's solution:  $\Lambda$  finite ring  $(|\Lambda|, p) = 1$

$\mathcal{X} =$  small  $v$ -stack

Def:  $\text{Set}(\mathcal{X}, \Lambda) = \{ \mathcal{F} \in \mathcal{D}(\mathcal{X}, \Lambda) \mid \forall S \xrightarrow{f} \mathcal{X} \text{ with } S \text{ strictly totally disconnected} \}$   
 $\Delta \text{ Set } \mathcal{D}(\mathcal{X}_a, \Lambda)$   $f^*\mathcal{F} \in \mathcal{D}(S_a, \Lambda)$



Strictly disconnected  $\Rightarrow D(S_{\text{ét}}, \Lambda) = D(|S|, \Lambda)$ .

$\rightarrow X = \text{locally spatial diamond}$   $D_{\text{ét}}(X, \Lambda) = D(X_{\text{ét}}, \Lambda)$

Morphisms:  $X \xrightarrow{f} Y$  representable in locally spatial diamonds.

$$D_{\text{ét}}(X, \Lambda) \begin{array}{c} \xrightarrow{f_*} \\ \xleftarrow{f^*} \end{array} D_{\text{ét}}(Y, \Lambda)$$

easy to define

$\exists D_{\text{ét}}(X, \Lambda)$   $\infty$ -stable category s.t.  $\pi_0 D_{\text{ét}}(X, \Lambda) = D_{\text{ét}}(X, \Lambda)$ .

$f: X \rightarrow Y$  representable in loc. spatial diamonds  
+ partially proper  
"overconvergent morphism"

$\rightarrow (\text{Bun}_G / \mathcal{Y}_{\text{pa}}(\mathbb{A}_q))$  is partially proper  $\text{Bun}_G(R, R^+) = \text{Bun}_G(R, R^0)$

$\rightarrow X/K$  separated rigid space

$X / \mathcal{Y}_{\text{pa}}(K)$  is p.p.  $\Leftrightarrow \forall U \subseteq X$  qc. open

$\overline{U}^X = \text{proper pseudo-adic space} / \mathcal{Y}_{\text{pa}}(K)$ .

$f: X \rightarrow Y \rightsquigarrow Rf_!: D_{\text{ét}}(X, \Lambda) \rightarrow D_{\text{ét}}(Y, \Lambda)$

$f: X \rightarrow Y$  pp.  
 $\uparrow$  loc. spatial diamonds

$Rf_! = \text{derived functor of } \Gamma_c(X/Y, \mathbb{F})$

$:= \{ s \in \mathcal{F}(X) \mid \text{Supp}(s) \xrightarrow{\text{proper}} Y \}$

$\rightarrow$  extends to  $f_!: D_{\text{ét}}(X, \Lambda) \rightarrow D_{\text{ét}}(Y, \Lambda)$   
thanks to proper base change.



Proposition - If  $\sup \deg(\text{qco fibers of } f) < +\infty$ , (4)

$(f: X \rightarrow Y)$   $f_!$  has a right adjoint  $f^!: \text{Det}(Y, \Lambda) \rightarrow \text{Det}(X, \Lambda)$ .

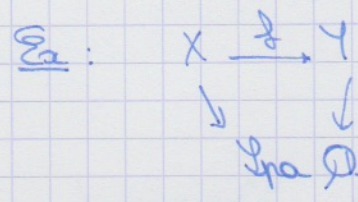
$\rightarrow$  called Lurie's adjunction.

Remark: Deligne in SGA 4: uses Godement resolutions.

Coh. smooth morphisms:  $X \xrightarrow{f} Y$

Def:  $f$  coho. smooth if  $\exists L \in \text{Det}(X, \Lambda)$  invertible

s.t.  $f^! = f^* \otimes L$ . ( $\rightarrow$  v. loc  $\Delta[m]$   $m \in \mathbb{Z}$ )



q.s. rigid spaces

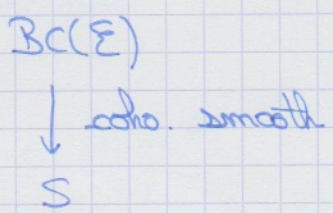
$f$  smooth  $\Rightarrow f^!$  coho. smooth

$\uparrow$   
Tulzer Serre duality.

Property - Coh. smoothness is v. local / base.

Ex:  $S \subset \mathbb{A}^2_{\mathbb{F}_q}$ ,  $\mathcal{E} = \text{v. b.} / X_S$ , fiberwise /  $S$   $\mathcal{E}$  has  $> 0$  slopes.

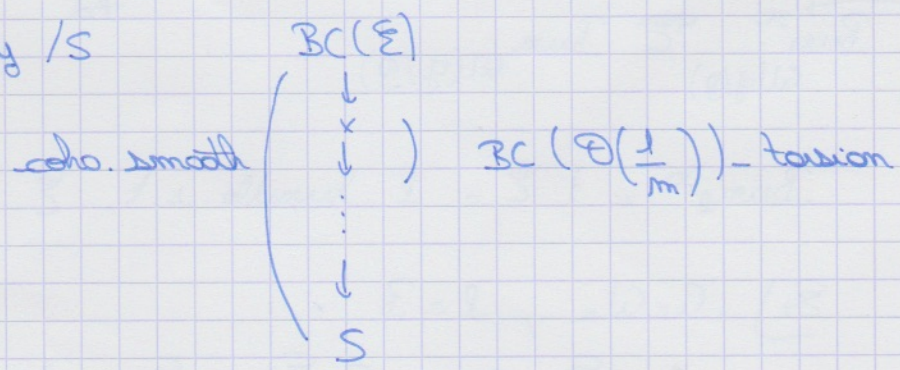
$$\text{BC}(\mathcal{E}) : T/S \rightarrow H^0(X_T, \mathcal{E}|_{X_T})$$



$\rightarrow$  prove that v. locally /  $S$ ,  $\mathcal{E}$  has a filtration by v. b., graded pieces  $\approx \mathcal{O}(\frac{d}{m})$ ,  $m \geq 1$ .

$$\text{BC}(\mathcal{O}(\frac{d}{m})) \approx \mathbb{B}_S^{m, 1/p^\infty}$$

v. locally /  $S$





last ex:

$X = \text{loc. spatial diamond} / S$

$K = \text{pro-p-group}$

$X \supset K$  free action / S

$\downarrow f$

S

X

a  $\downarrow$

$X/K$  ] loc. spatial

$X/S$  coho smooth

$\Rightarrow (X/K)/S$  coho. smooth

b  $\downarrow$

S

Ex:  $\Sigma$  v.b. /  $X_S$   $< 0$  slopes fiberrise / S

$R^1 T_* \Sigma = \text{sheaf associated to } T/S \mapsto H^1(X_T, \Sigma|_{X_T})$

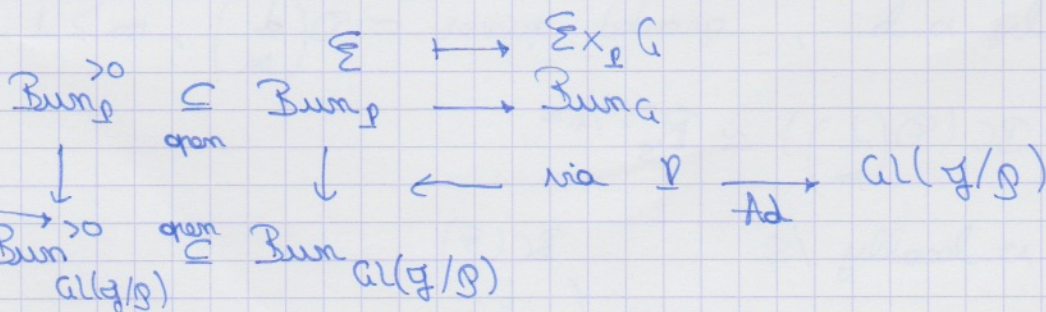
coho sm / S

$\rightarrow R^1 T_* \mathcal{O}(-1) = \mathcal{O}_a^\diamond / \underline{E} = \left( \mathcal{O}_a^\diamond / \mathcal{O}_{\underline{E}} \right) / \left( \underline{E} / \mathcal{O}_{\underline{E}} \right)$   
 coho sm.

Some coho. smooth charts on  $\text{Bun}_G$ :

$P$  parabolic subgroup of  $G$

$\mathfrak{g} = \text{lie } P, \mathfrak{h} = \text{lie } G$



HN slopes  $> 0$

$\text{Bun}_P^{>0} = \{ \Sigma = P\text{-bundle s.t. } \Sigma \times_P \mathfrak{g}/\mathfrak{p} \text{ has } > 0 \text{ slopes.} \}$

Ex  $G = \text{GL}_n, P = B$

"anti-HN"

$0 = \Sigma_0 \neq \dots \neq \Sigma_n = \Sigma$  s.t.  $\text{deg } \Sigma_i / \Sigma_{i-1} < \text{deg } \Sigma_{i+1} / \Sigma_i$



Thm  $Bun_G^{>0} \rightarrow Bun_G$  is coh. smooth. (5)

Application Fix  $b \in G(\check{E})$  s.t.  $v_b$  defined  $/ E$   
 $v_b: \mathbb{D} \rightarrow G$

$P = P_{v_b}$ ,  $\pi = G(v_b)$  base of  $P$ .

$b = b_\pi \in \pi(\check{E})$

Def:  $\mathcal{J}_b =$  moduli of  $P$ -bundles  $\Sigma$  s.t. geometrically fiberwise  $\Sigma_{x_i} \pi \simeq \Sigma_{b_\pi}$  (as an  $\pi$ -bundle).

$\mathcal{J}_b: \mathcal{J}_b \rightarrow Bun_G$   
 connected

Thm:  $\mathcal{J}_b: \mathcal{J}_b \rightarrow Bun_G$  is coh. smooth.

$\Rightarrow \mathcal{J}_b$  is open.

$[b] \in \mathcal{B}(G) = |Bun_G|$   
 $\text{Im}(\mathcal{J}_b)$

$\Rightarrow$  any generalization of  $[b]$  lies in  $\text{Im}(\mathcal{J}_b)$ .

$\Rightarrow Bun_G^{>0}$  is connected.

Ex:  $G = GL_n$  moduli of extensions  $0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{O}(1) \rightarrow 0$

$E = \mathcal{O}_n$

$R^1 T_x \mathcal{O}(-1) = \mathbb{A}^1 / \mathcal{O}_n$

$\mathcal{J}_b =$  moduli of ext.  $0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{L}' \rightarrow 0$   
 $\text{deg } 0 \quad \text{deg } 1$

$\mathcal{J}_b = \left[ \mathbb{A}^1 / \begin{pmatrix} \mathcal{O}_n^x & \mathcal{O}_n \\ 0 & \mathcal{O}_n^x \end{pmatrix} \right] \cong \left[ \mathbb{A}^1 / \begin{pmatrix} \mathcal{O}_n^x & \mathcal{O}_n \\ 0 & \mathcal{O}_n^x \end{pmatrix} \right]$

(0)  $\leftarrow$  locus where  $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{O}(1)$

$\leftarrow$  locus where  $\mathcal{E} \simeq \mathcal{O} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



$Bun_G, \mathbb{F}_q$   $K = \Lambda[0]$  dualizing complex on  $Bun_G \leftrightarrow \text{Dat}(Bun_G, \Lambda)$

Def -  $\mathcal{F} \in \text{Dat}(Bun_G, \Lambda)$  is reflexive if  $\mathcal{F} \simeq \mathbb{D}\mathcal{F}$   
 $\forall \mathcal{L} \in \mathcal{G}(\mathbb{F})$

$$j_b: \left[ \cdot / \underline{\text{Aut}}(\Sigma_b) \right] \xrightarrow{\text{loc. closed}} Bun_G$$

$$\parallel$$

$$\underline{\text{Aut}}(\Sigma_b)^\circ \times \underline{J}_b(\mathbb{F})$$

Thm (Deligne) -  $\mathcal{F} \in \text{Dat}(Bun_G, \Lambda)$  reflexive.  
 $\Leftrightarrow \forall \mathcal{L} \forall i \in \mathbb{Z} \mathcal{H}^i(j_b^* \mathcal{F})$  as a smooth repr. of  $\underline{J}_b(\mathbb{F})$  is admissible

sheaf of  $\Lambda$ -mod on  $\left[ \cdot / \underline{\text{Aut}}(\Sigma_b) \right]$

Corollary -  $\forall \mathcal{L} \pi$  smooth admissible repr. of  $\underline{J}_b(\mathbb{F})$   
 $j_b! \mathcal{F}_\pi$  and  $Rj_b^* \mathcal{F}_\pi$  are reflexive.  
 $\hookrightarrow$  local system  $\left[ \cdot / \underline{\text{Aut}}(\Sigma_b) \right]$

Proof [ satisfies the hypothesis of the thm. +  $\mathcal{F}$  reflexive  $\Rightarrow \mathbb{D}\mathcal{F}$  reflexive

Cor -  $\text{Bew}(Bun_G, \Lambda) = \{ \mathcal{F} \in \text{Dat}(Bun_G, \Lambda) \mid \mathcal{F} \text{ reflexive and } \forall \mathcal{L} \mathcal{H}^i(j_b^* \mathcal{F}) = 0 \text{ if } i > \langle \nu_b, \rho \rangle \}$   
 $\mathcal{H}^i(j_b! \mathcal{F}) = 0 \text{ if } i < \langle \nu_b, \rho \rangle$

is an abelian category -

Simple object :  $j_b! \mathcal{F}_\pi$   $\pi = \text{smooth mod of } \underline{J}_b(\mathbb{F})$   
 $\text{Im}(\pi_j! \rightarrow \pi Rj_b^*)$