11/06/2018 Fargues, Geometrization Conjecture of L.L. correspondence. (1)

Purpose: * Explain the conjecture:

$$\left\{\begin{array}{l}\text{Local Langlands}\\ \text{parameters for } G/\mathbb{Q}_p\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Perverse sheaves}\\ \text{on } \mathrm{Bun}_G\end{array}\right\}$$

Bun_G: Perfectoid stack of G-bundles / curve

$$\underbrace{\varphi}_{} \longmapsto \mathcal{F}_\varphi / \mathrm{Bun}_G$$

L. parameter s.t.

$\forall b \in G(\breve{\mathbb{Q}}_p)$ basic $j_b : [\cdot / J_b(\mathbb{Q}_p)] \xrightarrow{\text{open}} \mathrm{Bun}_G$

$\varphi \longmapsto j_b^* \mathcal{F}_\varphi$ gives classical local Langlands.

* Explain some ideas about the construction of local Langlands parameters:

$\pi \longmapsto \varphi_\pi$ (in progress with P. Scholze)

using perverse sheaves on Bun_G.

For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell\}$ define a $\widehat{G}$-pseudochar from $W_{\mathbb{Q}_p}$ with values in $Z(\mathrm{Perv}(\mathrm{Bun}_G, \Lambda))$.

For $\pi \in \mathrm{Irr}(J_b(\mathbb{Q}_p), \Lambda)$, b basic:

$j_{b\,!*} \mathcal{F}_\pi$ = irr. perverse on Bun_G (local syst. on $[\cdot/J_b(\mathbb{Q}_p)]$)

$\hookrightarrow$ char. $Z(\_) \twoheadrightarrow \Lambda \rightsquigarrow \varphi_\pi$ up to semi-simplification.

§. The fundamental curve of p-adic Hodge theory.

E local field $\longrightarrow \mathbb{F}_q((\pi))$

$\searrow [E : \mathbb{Q}_p] < +\infty$, $\mathbb{F}_q = \mathcal{O}_E/\pi$

$F/\mathbb{F}_q$ perfectoid field $\mathrm{Spa}\, F$

* $E = \mathbb{F}_q((\pi))$ "Hartl-Pink case"

$Y_F = \mathbb{D}_F^* = \{0 < |\pi| < 1\} \subseteq \mathbb{A}_F^1$

$$\mathcal{O}(Y_F) = \left\{ \sum_{n \in \mathbb{Z}} x_n \pi^n \;\middle|\; x_n \in F,\ \forall \rho \in]0,1[,\ \lim_{|n| \to +\infty} |x_n| \rho^n = 0 \right\}$$

loc. of finite type → $\mathbb{D}^*_F$ ← not locally of finite type

$$\begin{tikzcd} & \mathbb{D}^*_F \arrow[dl] \arrow[dr] & \\ \mathrm{Spa}\, F & & \mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}\, \mathbb{F}_q((\pi)) \end{tikzcd}$$

$$\mathbb{F}_q((\pi)) \subseteq \mathcal{O}(Y_F).$$

* $E/\mathbb{Q}_p$ $\quad \underbrace{A}_{\text{Fontaine's Ainf}} = W_{\mathcal{O}_E}(\mathcal{O}_F) = \{ \sum_{n \geq 0} [x_n] \pi^n \mid x_n \in \mathcal{O}_F \}.$

$$\underbrace{A\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right]}_{} = \left\{ \sum_{n \gg -\infty} [x_n] \pi^n \;\middle|\; \begin{array}{l} x_n \in F \\ \sup |x_n| < \infty \end{array} \right\} \quad \begin{array}{l} 0 < |\varpi| < 1 \\ \varpi \in F \end{array}$$

holo. fonctions on Y_F mero. along (π) and $([\varpi])$.

$\rho \in]0,1[\qquad |\sum_n [x_n] \pi^n|_\rho = \sup_n |x_n| \rho^n$

$|.|_\rho$ = Gauss norm, radius ρ.

$\mathcal{O}(Y_F) :=$ Fréchet E-alg. completion of $A\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right]$ w.r.t. $(|.|_\rho)_{\rho \in]0,1[}$.

$Y_F = \mathrm{Spa}(A, A) \setminus V(\pi [\varpi])$

with the $(\pi, [\varpi])$-adic topo.

E-adic "Stein" space → completely determined by $\mathcal{O}(Y_F)$.

<u>The adic curve</u>: $\quad A_F$

$\varphi = \mathrm{Frob}_F \qquad (q\text{-Frob})$

↳ induces an automorphism of Y_F

$$\varphi\left(\sum_n [x_n] \pi^n\right) = \sum_n [x_n^q] \pi^n$$

$Y_F \circlearrowleft \varphi$ properly discontinuous without fixed points.

$\rho \in]0,1[\qquad \varphi(\text{annulus } \{|\pi| = \rho\}) = \text{annulus } \{|\pi| = \rho^{1/q}\}$

<u>Def</u>. $X_F = Y_F / \varphi^{\mathbb{Z}}$ $\qquad$ q.c. E-adic space not of finite type

[Prop. $\mathrm{Spa}(A, A) \setminus V(\pi[\varpi])$ is sheafy.]

Classical points.

Def. $\xi = \sum_{n \geq 0} [x_n] \pi^n \in \underset{\mathcal{O}(Y_F)^+}{\overset{\shortparallel}{A_F}}$ is primitive of deg $d \geq 1$ if $x_0 \neq 0$, $x_0, \ldots, x_{d-1} \in \mathfrak{m}_F$ and $x_d \in \mathcal{O}_F^\times$.

comes from Weierstrass factorization theory.

Rmk) deg $d \times$ deg d' = deg $d+d'$

Thm. ξ primitive irreducible of deg $d \geq 1$.

(1) $K = \mathcal{O}(Y_F)/\xi$ perfectoid field $/E$ with

$$\begin{cases} F \hookrightarrow K^\flat \\ x \mapsto ([x^{q^{-n}}] \bmod \xi)_{n \geq 0} \end{cases}$$ satisfies $[K^\flat : F] = d$.

(2) F alg. closed then $d = 1$.

$\longrightarrow$ Weierstrass factorization $\forall \xi$ primitive deg d.

$$\xi = \underset{A^\times}{\overset{\cap}{u}} \times \underbrace{(\pi - [a_1]) \ldots (\pi - [a_d])}_{\text{not unique}}$$

(3) $\forall I = [\rho_1, \rho_2] \subseteq]0, 1[$.

Y_I = compact annulus $\{ |\pi| \in I \}$.

$\mathcal{O}(Y_I)$ = PID with $\underset{\{(\xi) \mid \xi \text{ primitive irreducible}\}}{\overset{\shortparallel}{\mathrm{Spm}(\mathcal{O}(Y_I))}}$

+ good theory of Newton polygons for elements of $\mathcal{O}(Y_F)$.

Def. $\underset{|Y_F|}{\overset{\cap}{|Y_F|^{cl}}} = \{ V(\xi) \mid \xi \text{ primitive irreducible} \}$

$|X_F|^{cl} = |Y_F|^{cl} / \varphi^{\mathbb{Z}}$.

The schematical curve.

$\mathcal{O}(1)$ line bundle $/ X_F$ $\qquad X_F = Y_F / \varphi^{\mathbb{Z}}$

Trivial on Y_F, automorphy factor $\pi^{-1}\varphi$.

$d \in \mathbb{Z}$ $\quad H^0(X_F, \mathcal{O}(d)) = \begin{cases} 0 & \text{if } d < 0 \\ E & \text{if } d = 0 \\ \mathcal{O}(Y_F)^{\varphi = \pi^d} \end{cases}$

(∞-dim. E-Banach space if $d > 0$

Def. $\mathcal{X}_F = \mathrm{Proj}\big(\underbrace{\bigoplus_{d \geq 0} \mathcal{O}(Y_F)^{\varphi = \pi^d}}_{\text{graded alg. of Fontaine's periods.}} \big)$

$\rightarrow$ E-scheme not of finite type.

Thm (1) $\mathcal{X}_F$ is a Dedekind E-scheme.

(2) $\exists$ morphism of ringed spaces $X_F \longrightarrow \mathcal{X}_F$ $\uparrow$ GAGA type

$|X_F|^{cl} \xrightarrow{\sim} |\mathcal{X}_F| = \text{closed point}$

s.t. if $\tilde{x} \longmapsto x$ $\quad k(\tilde{x}) = k(x) = \text{perfectoid} / E$.

$\widehat{\mathcal{O}}_{\mathcal{X}_F, x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X_F, \tilde{x}} = B^+_{dR}(\underbrace{k(x)}_{\text{perfectoid}}) \xrightarrow{\theta} k(x)$

(3) $\forall f \in E(\mathcal{X}_F)^\times$, $\deg(\mathrm{div} f) = 0$ $\quad$ $\left[\deg x = [k(x)^\flat : F] \right]$

$\rightarrow$ good notion of degree of a v.b. $\rightarrow$ (Harder-Narasimhan filtrations.

(4) If F alg. closed $\forall x \in |\mathcal{X}_F|$ $\deg x = 1$.

If $t \in H^0(\mathcal{O}(1)) \setminus \{0\}$ $\quad V^+(t) = \{\infty_t\}$

$\mathcal{X}_F \setminus \{\infty\} = \mathrm{Spec}\big(\underbrace{\mathcal{O}(Y_F)\left[\tfrac{1}{t}\right]^{\varphi = Id}}_{\text{PID.}} \big)$

$\left(\mathcal{O}(Y_F)\left[\tfrac{1}{t}\right]^{\varphi = Id}, -\mathrm{ord}\, \infty_t\right)$ not euclidean

$H^1(\mathcal{O}(-1)) \neq 0$

(contrary to $\mathbb{P}^1$ $\quad (k[t], \deg)$.

(5) GAGA v.b.$/_{\mathcal{X}_F} \xrightarrow{\sim}$ v.b.$/_{X_F}$ (Kedlaya).

Picard group. F alg. closed

$$\deg : \underset{\underset{\langle \mathcal{O}(1)\rangle}{\|}}{\mathrm{Pic}(X_F)} \xrightarrow{\sim} \mathbb{Z}$$

Classification of vector bundles:

F alg. closed

$n \geq 1$ $\quad E_n / E$ deg n unramified extension.

$$\mathbb{Z}/n\mathbb{Z} \left(\begin{array}{ccc} Y/_{\varphi^{n\mathbb{Z}}} & = X_{F,E_n} & = X_{F,E} \otimes_E E_n \\ \downarrow \pi_n & \downarrow & \\ Y/_{\varphi} & = X_{F,E} & \end{array} \right.$$

$\lambda = \frac{d}{h}$ $\quad (d,h) = 1$ $\qquad \underbrace{\mathcal{O}(\lambda)} = \pi_{h*} \mathcal{O}_{X_{F,E_h}}(d)$

stable slope λ vector bundle.

Thm. (1) Any slope λ s.s. vector bundle $/X_F$ is isomorphic to $\mathcal{O}(\lambda)^m$ for some m.

(2) The Harder-Narasimhan filtration is split.

(3)
$$\left(\begin{array}{ccc} \left\{ \lambda_1 \geq \ldots \geq \lambda_n \;\middle|\; \begin{array}{c} n \in \mathbb{N} \\ \lambda_i \in \mathbb{Q} \end{array} \right\} & \xrightarrow{\sim} & \mathrm{Bun}_{X_F}/\sim \\ (\lambda_1, \ldots, \lambda_n) & \longmapsto & [\bigoplus_i \mathcal{O}(\lambda_i)] \end{array} \right.$$

$\rightarrow$ difficult : uses periods of p-div groups.

Ex) Any modification $0 \to \mathcal{E} \to \mathcal{O}\left(\frac{1}{n}\right) \to \underbrace{\mathcal{F}}_{\text{torsion deg } 1} \to 0$

is such that $\mathcal{E} \cong \mathcal{O}^n$

$\Leftrightarrow$ the Gross-Hopkins period morphism

Lubin-Tate $\xrightarrow[\text{morphism}]{\text{period}}$ $\mathbb{P}^{n-1}$ is surjective.

Classification of G-bundles.

G/E reductive group $\quad \mathbb{F}$ alg. closed.

$\breve{E} = \widehat{E^{un}} \circlearrowleft \sigma : x \mapsto x^q$ ← semi-lin. auto.

$\varphi\text{-Mod}_{\breve{E}}$ = isocrystals = $\{(D,\varphi)\}$

finite $\breve{E}$-vs

$$\begin{cases} \varphi\text{-Mod}_{\breve{E}} \longrightarrow \mathrm{Bun}_{X_{\mathbb{F}}} & \text{essentially surjective} \\ (D,\varphi) \longmapsto Y_{\mathbb{F}} \times_{\varphi^{\mathbb{Z}}} D =: \mathcal{E}(D,\varphi) \end{cases}$$

$b \in G(\breve{E}) \quad \mathcal{E}_b : \begin{cases} \mathrm{Rep}(G) \xrightarrow{\otimes} \text{Isocrystals} \xrightarrow{\mathcal{E}(-)} \mathrm{Bun}_{X_{\mathbb{F}}} \\ (V,\rho) \longmapsto (V_{\breve{E}}, \rho(b)\sigma) \end{cases}$

G-bundle

$\mathcal{E}_b$ ⟶ Tannakian sense

⟶ G-torsor / $X_{\mathbb{F}}$

$$\mathcal{T} \downarrow X_{\mathbb{F}} \quad \circlearrowleft G$$

Thm (F., Anschutz): $\begin{cases} B(G) \xrightarrow{\sim} H^1_{ét}(X_{\mathbb{F}}, G^{\mathcal{E}}) \\ [b] \longmapsto [\mathcal{E}_b] \end{cases}$

+ properties: $\ast$ $H^1(E,G) \subset B(G)$

induced by $H^1_{ét}(\mathrm{Spec}\,E, G) \longrightarrow H^1_{ét}(X_{\mathbb{F}}, G)$.

$\ast$ b basic $\Leftrightarrow$ $\mathcal{E}_b$ is semi-stable

in Atiyah-Bott sense.

longest element in Weyl-group

$\ast$ $-w_0.[\nu_b]$ = HN polygon of $\mathcal{E}_b$

Newton polygon $\quad \in$ (positive Weyl chamber)

$\ast$ $\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma$ ← $\mathrm{Gal}(\bar{E}/E)$

can be interpreted as the $-c_1$ of a G-bundle

<u>The curve in family</u>. E as before $\to \mathbb{F}_q((\pi))$
$\searrow [E:\mathbb{Q}_p]<\infty$.

$\mathrm{Perf}_{\mathbb{F}_q} \ni \underset{\mathrm{Spa}\,F}{\overset{\|}{S}} \longmapsto X_S = E\text{-adic space functorial in } S$
$=$ "family of curves $(X_{k(s)})_{s\in S}$"

* $E = \mathbb{F}_q((\pi))$ $\quad \varphi = \mathrm{Frob}_S \circlearrowright Y_S = \mathbb{D}_S^* = \{0<|\pi|<1\} \subseteq \mathbb{A}^1_S$
$\searrow \mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}\,E$

$$X_S = Y_S/\varphi^{\mathbb{Z}}.$$

* $E/\mathbb{Q}_p$ $\quad S = \mathrm{Spa}(R,R^+)$ $\quad \mathbb{A}_{R,R^+} = W_{\mathcal{O}_E}(R^+)$
$\varphi \circlearrowright Y_S = \mathrm{Spa}(\mathbb{A}_{R,R^+}, \mathbb{A}_{R,R^+}) \setminus V(\pi[\varpi])$
$\varpi \in R^{\circ\circ} \cap R^\times$

$$X_S = Y_S/\varphi^{\mathbb{Z}}.$$

<u>Div^1</u>: object that will replace the curve in a cohomological context $\mathrm{Bun}_G \times \mathrm{Div}^1$.

Recall $\mathrm{Spa}(E)^\diamond$ = v-sheaf of untilts over E
$\in \widetilde{\mathrm{Perf}}_{\mathbb{F}_q}$

$\mathrm{Spa}(E)^\diamond(S) = \{(S^\#, \iota) \mid S^\# \in \mathrm{Perf}_E,\ \iota: S \xrightarrow{\sim} S^{\#,\flat}\}/\sim$

Quickly said "$S^\#$ is an untilt of S".

<u>Ex)</u> $E = \mathbb{F}_q((\pi))$ $\quad \mathrm{Spa}(E)^\diamond$ represented by
$\mathrm{Spa}(\mathbb{F}_q((\pi^{1/p^\infty})))$

<u>Rmk</u>. $\widetilde{\mathrm{Perf}}_E = \widetilde{\mathrm{Perf}}_{\mathbb{F}_q} / \mathrm{Spa}(E)^\diamond$.

$S = \mathrm{Spa}(R,R^+)$ aff. perf.
$\xi = \sum_{n\geq 0} [x_n]\pi^n \in \mathbb{A}_{R,R^\circ} = \mathcal{O}(Y_{R,R^\circ})^+$ is primitive of degree 1 if $x_0 \in R^{\circ\circ} \cap R^\times$, $x_1 \in (R^\circ)^\times$

Thm - (1) If $R^\#$ = untilt of R, $\theta : A_{R,R^\circ} \twoheadrightarrow R^{\#,\circ}$ is surjective and $\ker\theta = (\xi)$, ξ primitive deg. 1

(2) Reciprocally if $\xi \in A_{R,R^\circ}$ deg. 1 primitive then $A_{R,R^\circ}\left[\frac{1}{\pi}\right] / (\xi)$ is an untilt of R.

$\Rightarrow$ Untilts of $S \in \mathrm{Perf}_{\mathbb{F}_q}$
$\quad\parallel$
Cartier divisors

$$\begin{tikzcd} S^\# \arrow[r, hook] \arrow[dr] & Y_S \arrow[d] \\ & \mathrm{Spa}\, E \end{tikzcd}$$

locally $V(\xi)$ with ξ deg 1 primitive

The construction $\mathrm{Spa}(R,R^+) \longmapsto \mathrm{Spa}(A_{R,R^+}, A_{R,R^+}) \setminus V([\varpi])$ glues to a construction $S \longmapsto \mathcal{Y}_S = \mathcal{O}_E$ - adic space, $S \in \mathrm{Perf}_{\mathbb{F}_q}$

$$S \xhookrightarrow{\pi=0} \mathcal{Y}_S \underset{\pi\neq 0}{\hookleftarrow} Y_S \hookleftarrow S^\#$$

" Untilt of S = deformation of the divisor $\{\pi = 0\}$ in $\mathcal{Y}_S$ to a divisor in $\{\pi \neq 0\}$ ".

Div^1 : $S \in \mathrm{Perf}_{\mathbb{F}_q}$ $\quad \mathrm{Div}^1 \in \widetilde{\mathrm{Perf}}_{\mathbb{F}_q}$

$$\mathrm{Div}^1(S) = \left\{ (\mathcal{L}, u) \;\middle|\; \begin{array}{l} \mathcal{L} \text{ line bundle} / X_S \\ u \in H^0(X_S, \mathcal{L}), \\ \forall s \in S \;\; u|_{X_{k(s),k(s)^+}} \neq 0 \\ \text{and s.t. } \forall \bar{s} \to S \;\; \mathcal{L}|_{X_{k(s),k(s)^+}} \text{ has deg 1.} \end{array} \right\} \Big/ \sim$$

The preceding construction defines $(\mathrm{Spa}\, E)^\diamond \longrightarrow \mathrm{Div}^1$ via

$$\begin{tikzcd} S^\# \arrow[r, hook] \arrow[dr, "\text{Cartier div.}"'] & Y_S \arrow[d] \\ & X_S \end{tikzcd}$$

$$\begin{tikzcd} (\mathrm{Spa}\, E)^\diamond \arrow[r] \arrow[d] & \mathrm{Div}^1 \\ (\mathrm{Spa}\, E)^\diamond / \varphi^{\mathbb{Z}} \arrow[ur, "\exists \text{ factorization}"'] & \end{tikzcd}$$

Proposition. $\mathrm{Spa}(E)^\diamond / \varphi^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Div}^1$.

$\longrightarrow$ Div^1 = q.c. diamond

⚠ This is not spatial, not quasi-separated

$$\mathrm{Spa}(E)^\diamond \times_{\mathrm{Div}^1} \mathrm{Spa}(E)^\diamond = \coprod_{\mathbb{Z}} \mathrm{Spa}(E)^\diamond$$

$$\begin{array}{l} \mathrm{Div}^1 \\ \downarrow \\ \mathrm{Spa}\,\mathbb{F}_q \\ \uparrow \end{array}$$

representable in proper coho. smooth. diamonds

$\mathrm{Spa}\,\mathbb{F}_q$) final object in $\widetilde{\mathrm{Perf}}_{\mathbb{F}_q}$

is not quasi. separated

(Rmk $\quad \mathrm{Spa}\,\mathbb{F}_q((x^{1/p^\infty})) \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\,\mathbb{F}_q((y^{1/p^\infty}))$

$= \mathrm{Spa}(\mathbb{F}_q[[x^{1/p^\infty}, y^{1/p^\infty}]]) \setminus V(xy)$