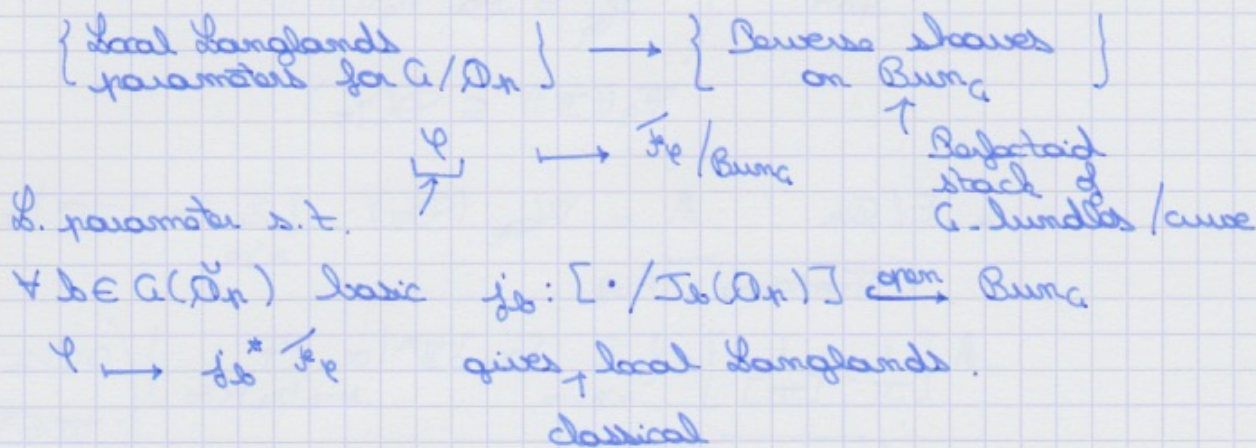


11/06/2018 Faugus, Geometrization Conjecture of L.L. correspondence. (1)

Purpose: * Explain the conjecture:



* Explain some ideas about the construction of local Langlands parameters:

$$\mathcal{T} \mapsto \mathcal{Y}_{\mathcal{T}} \quad (\text{in progress with P. Scholze})$$

using reverse sheaves on Bun_G .

For $\Lambda \in \{\overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_p\}$ define a \widehat{G} -pseudochar from W_{an} with values in $\mathcal{Z}(\text{Bun}_G, \Lambda)$.

For $\mathcal{T} \in \text{Irr}(J_b(\mathbb{Q}_p), \Lambda)$, b basic:

$$\begin{array}{ccc}
 \text{local syst.} & \text{fib}^* \overline{J}_p = \text{via reverse on } \text{Bun}_G & \\
 \text{on } [\cdot / J_b(\mathbb{Q}_p)] & \xrightarrow{\quad} & \text{char. } \mathcal{Z}(\cdot) \xrightarrow{\Lambda} \mathcal{Y}_{\mathcal{T}} \text{ up to} \\
 & & \text{semi-simplification.}
 \end{array}$$

§. The fundamental case of p-adic Hodge theory.

$$E \text{ local field} \longrightarrow \mathbb{F}_q((\mathcal{T}))$$

$$\searrow [E: \mathbb{Q}_p] < +\infty, \mathbb{F}_p = \mathcal{O}_E / \mathcal{T}$$

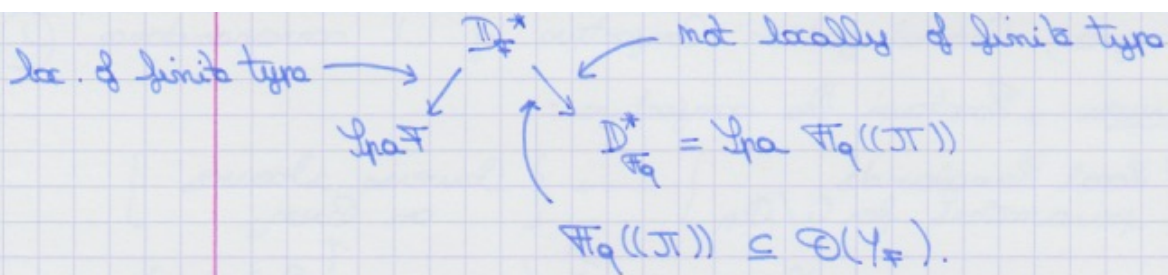
$$\mathbb{F} / \mathbb{F}_q \text{ perfectoid field} \quad \text{Spa } \mathbb{F}$$

$$* E = \mathbb{F}_q((\mathcal{T}))$$

"Hartl - Pink case"

$$\mathcal{Y}_{\mathbb{F}} = \mathbb{D}_{\mathbb{F}}^* = \{ 0 < |\mathcal{T}| < 1 \} \subseteq A_{\mathbb{F}}^{\pm}$$

$$\mathcal{O}(\mathcal{Y}_{\mathbb{F}}) = \left\{ \sum_{m \in \mathbb{Z}} x_m \mathcal{T}^m \mid x_m \in \mathbb{F}, \forall \rho \in]0, 1[\lim_{|m| \rightarrow \infty} |x_m| \rho^m = 0 \right\}$$



* E/\mathcal{O}_p $A = \widehat{W_{\mathcal{O}_E}(\mathcal{O}_F)} = \left\{ \sum_{n \geq 0} [x_n] \mathcal{J}^n \mid x_n \in \mathcal{O}_F \right\}$.

Fontaine's A_{inf}

$A \left[\frac{1}{\mathcal{J}}, \frac{1}{[\omega]} \right] = \left\{ \sum_{n \geq -\infty} [x_n] \mathcal{J}^n \mid x_n \in \mathcal{F}, \sup |x_n| < \infty \right\}$

$\left(\begin{array}{l} 0 < |\omega| < 1 \\ \omega \in \mathcal{F} \end{array} \right)$

Ado. functions on Y_F zero along (\mathcal{J}) and $([\omega])$.

$\rho \in]0, 1[\quad \left| \sum_n [x_n] \mathcal{J}^n \right|_\rho = \sup_n |x_n| \rho^n$

l. $|\cdot|_\rho =$ Gauss norm, radius ρ .

$\mathcal{O}(Y_F) :=$ Fréchet E -alg. completion of $A \left[\frac{1}{\mathcal{J}}, \frac{1}{[\omega]} \right]$
w.r.t. $(|\cdot|_\rho)_{\rho \in]0, 1[}$.

$Y_F = \text{Spa}(A, A) \setminus V(\mathcal{J}, [\omega])$

\uparrow

with the $(\mathcal{J}, [\omega])$ -adic topo.

E -adic "Stein" space \rightarrow completely determined by $\mathcal{O}(Y_F)$.

The adic curve:

A_F

$\varphi = \text{Frob}_F \quad (q\text{-Frob})$

\hookrightarrow induces an automorphism of Y_F

$\varphi \left(\sum_n [x_n] \mathcal{J}^n \right) = \sum_n [x_n^q] \mathcal{J}^n$

$Y_F \curvearrowright \varphi$ properly discontinuous without fixed points.

$\rho \in]0, 1[\quad \varphi(\text{annulus } \{|\mathcal{J}| = \rho\}) = \text{annulus } \{|\mathcal{J}| = \rho^{1/q}\}$

Sol. $X_F = Y_F / \varphi^2$

q.c. E -adic space

not of finite type

[Prop. $\text{Spa}(A, A) \setminus V(\mathcal{J}[A])$ is shafly.]

(2)

Classical points.

$\mathcal{O}(Y_{\mathbb{F}})^{\times}$

Def. $\mathcal{J} = \sum_{n \geq 0} [x_n] \mathcal{J}^n \in \mathbb{A}_{\mathbb{F}}$ is primitive of $\text{deg } d \geq 1$ if $x_0 \neq 0$, $x_0, \dots, x_{d-1} \in m_{\mathbb{F}}$ and $x_d \in \mathcal{O}_d^{\times}$.

comes from Weierstrass factorization theory.

Prop. $\text{deg } d \times \text{deg } d' = \text{deg } d + d'$

Thm. \mathcal{J} primitive irreducible of $\text{deg } d \geq 1$.

(1) $K = \mathcal{O}(Y_{\mathbb{F}}) / \mathcal{J}$ perfectoid field / E with

$\mathbb{F} \hookrightarrow K^{\flat}$ satisfies $[K^{\flat} : \mathbb{F}] = d$.

$x \mapsto ([x^{p^{-n}}] \text{ mod } \mathcal{J})_{n \geq 0}$

(2) \mathbb{F} alg. closed then $d = 1$.

\rightarrow Weierstrass factorization $\forall \mathcal{J}$ primitive $\text{deg } d$.

$$\mathcal{J} = \mu \cdot \prod_{i=1}^m (\mathcal{J} - [a_i]) \dots (\mathcal{J} - [a_d])$$

not unique

(3) $\forall I = [e_1, e_2] \subseteq]0, 1[$.

$Y_I =$ compact annulus $\{ |T| \in I \}$.

$\mathcal{O}(Y_I) = \text{PID}$ with $\text{prim}(\mathcal{O}(Y_I))$

$\{ (\mathcal{J}) \mid \mathcal{J} \text{ primitive irreducible} \}$

+ good theory of Newton polygons for elements of $\mathcal{O}(Y_{\mathbb{F}})$.

Def. $|Y_{\mathbb{F}}|^d = \{ V(\mathcal{J}) \mid \mathcal{J} \text{ primitive irreducible} \}$

$|Y_{\mathbb{F}}|$

$$|X_{\mathbb{F}}|^d = |Y_{\mathbb{F}}|^d / \varphi_{\mathbb{F}}$$

The schematic curve.

$\mathcal{O}(1)$ line bundle / X_F

$$X_F = Y_F / \varphi^2$$

Critical on Y_F , automorphism factor $\pi^{-1}\varphi$.

$$d \in \mathbb{Z} \quad H^0(X_F, \mathcal{O}(d)) = \begin{cases} 0 & \text{if } d < 0 \\ E & \text{if } d = 0 \\ \mathcal{O}(Y_F)^{\varphi = \pi^d} & \text{if } d > 0 \end{cases}$$

∞ -dim. E -Banach space if $d > 0$

Def. $X_F = \text{Proj} \left(\bigoplus_{d \geq 0} \mathcal{O}(Y_F)^{\varphi = \pi^d} \right)$

graded alg. of Fontaine's periods.

$\rightarrow E$ -scheme not of finite type.

Thm. (1) X_F is a Godefrid E -scheme.

(2) \exists morphism of rigid spaces $X_F \rightarrow X_F$

$$|X_F|^{cl} \xrightarrow{\sim} |X_F| = \text{closed point}$$

\uparrow GAGA type

s.t. if $\tilde{x} \mapsto x \quad k(\tilde{x}) = k(x) = \text{perfectoid } / E.$

$$\widehat{\mathcal{O}}_{X_F, x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X_F, \tilde{x}} = \mathbb{B}_{dR}^+(k(x)) \xrightarrow{\Theta} k(x)$$

perfectoid

$$\begin{matrix} \nearrow \\ \text{[deg } x \\ = [k(x)^b : F] \end{matrix}$$

(3) $\forall f \in E(X_F)^{\times}, \text{deg}(\text{div } f) = 0$

\rightarrow good notion of degree of a v.l.

\rightarrow Harder-Narasimhan filtrations.

(4) $\exists f \in F$ alg. closed $\forall x \in |X_F| \text{ deg } x = 1.$

$\exists f \in H^0(\mathcal{O}(1)) \setminus \{0\} \quad V^+(f) = \{\infty_f\}$

$$X_F \setminus \{\infty_f\} = \text{Spa} \left(\mathcal{O}(Y_F) \left[\frac{\cdot}{f} \right]^{\varphi = \pi^d} \right)$$

PID.

$(\mathcal{O}(Y_F) \left[\frac{\cdot}{f} \right]^{\varphi = \pi^d}, \text{-ord } \infty_f)$ not euclidean

$$H^1(\mathcal{O}(-1)) \neq 0$$

(contrary to $\mathbb{P}^1(k[t], \text{deg}).$)

(5) GAGA v.b. / \mathbb{F} $\xrightarrow{\sim}$ v.b. / $X_{\mathbb{F}}$ (Kodaira) - (3)

Picard group - \mathbb{F} alg. closed

$$\text{deg}: \text{Pic}(X_{\mathbb{F}}) \xrightarrow{\sim} \mathbb{Z}$$

$$\parallel$$

$$\langle \mathcal{O}(1) \rangle$$

Classification of vector bundles:

\mathbb{F} alg. closed

$m \geq d$ E_m / E deg m unramified extension.

$$\mathbb{Z}/m\mathbb{Z} \left(\begin{array}{l} Y/p^m\mathbb{Z} = X_{\mathbb{F}, E_m} = X_{\mathbb{F}, E} \otimes_E E_m \\ \downarrow \mathcal{J}_m \\ Y/p\mathbb{Z} = X_{\mathbb{F}, E} \end{array} \right.$$

$$\lambda = \frac{d}{h} \quad (d, h) = 1 \quad \mathcal{O}(\lambda) = \mathcal{J}_d * \mathcal{O}_{X_{\mathbb{F}, E_d}}(d)$$

stable slope λ vector bundle.

Thm. (1) Any slope λ s.s. vector bundle / $X_{\mathbb{F}}$ is isomorphic to $\mathcal{O}(\lambda)^m$ for some m .

(2) The Harder-Narasimhan filtration is split.

$$(3) \left(\begin{array}{l} \{ \lambda_1 \geq \dots \geq \lambda_m \mid m \in \mathbb{N} \\ \lambda_i \in \mathbb{Q} \} \xrightarrow{\sim} \text{Bun}_{X_{\mathbb{F}}} / \sim \\ (\lambda_1, \dots, \lambda_m) \longmapsto [\bigoplus_i \mathcal{O}(\lambda_i)] \end{array} \right.$$

→ difficult: uses periods of p -div groups.

Ex) Any modification $0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}(\frac{1}{m}) \rightarrow \mathcal{F} \rightarrow 0$
 is such that $\mathcal{E} \cong \mathcal{O}^m$ \mathcal{F} torsion deg 1

⇔ The Gross-Hopkins period morphism

Lubin-Tate period morphism \mathbb{P}^{m-1} is surjective.

Classification of G-bundles

G/E reductive group \mathbb{F} alg. closed.

$$\tilde{E} = \widehat{E}^{un} \hookrightarrow \sigma: x \mapsto x^q$$

$\varphi: \text{Isoc}_{\tilde{E}}^v = \text{isocrystals} = \{(\mathcal{D}, \varphi)\}$ ← semi-lin. auto.

↖ limits \tilde{E}^v

$(\varphi: \text{Isoc}_{\tilde{E}}^v \rightarrow \text{Bun}_{X_{\mathbb{F}}})$ essentially surjective

$$(\mathcal{D}, \varphi) \mapsto \gamma_{\mathbb{F}, \varphi} \mathcal{D} =: \mathcal{E}(\mathcal{D}, \varphi)$$

$$\begin{array}{ccccc} b \in G(\tilde{E}) & \Sigma_b & \text{Rep}(G) & \xrightarrow{\otimes} & \text{Isocrystals} & \xrightarrow{\mathcal{E}(-)} & \text{Bun}_{X_{\mathbb{F}}} \\ G\text{-bundle} & \uparrow & (V, \rho) & \mapsto & (V_{\tilde{E}}^v, \rho(b)\sigma) & & \end{array}$$

Σ_b ↘ Tannakian sense
↘ G-torsor / $X_{\mathbb{F}}$

$$\begin{array}{c} T \\ \downarrow \\ X_{\mathbb{F}} \end{array} \Big) G$$

Thm (F., Anschütz): $\begin{cases} B(G) \xrightarrow{\sim} H_{\text{ét}}^1(X_{\mathbb{F}}, G^{\mathbb{F}}) \\ [\mathcal{D}] \mapsto [\Sigma_b] \end{cases}$

- + properties:
- * $H^1(E, G) \subset B(G)$
 - induced by $H_{\text{ét}}^1(\text{Spec } \tilde{E}, G) \rightarrow H_{\text{ét}}^1(X_{\mathbb{F}}, G)$.
 - * \mathcal{D} basic $\Leftrightarrow \Sigma_b$ is semi-stable in Atiyah-Bott sense.

longest element in Weyl-group

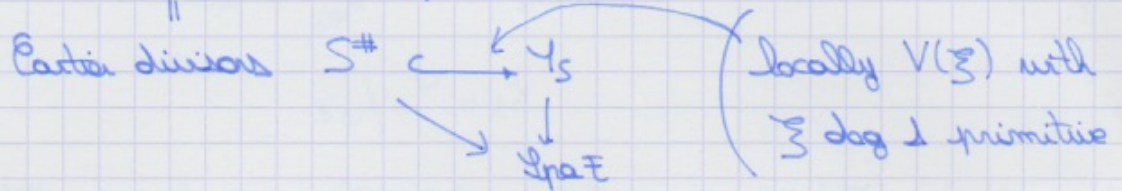
* $-w \cdot [\nu_{\mathcal{D}}] = \#N$ polygon of $\Sigma_b \in (\text{positive Weyl chamber})$

↖ Newton polygons

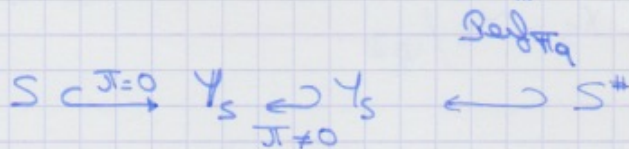
* $\kappa: B(G) \rightarrow \pi_1(G)_{\Gamma} \xleftarrow{\text{Gal}(\tilde{E}/\mathbb{F})}$
can be interpreted as the $-C_1$ of a G-bundle

Thm - (1) If $R^\# = \text{untilt of } R$, $\Theta: A_{R,R^\#} \rightarrow R^{\#0}$ is surjective and $\ker \Theta = (\xi)$, ξ primitive deg. 1
 (2) Reciprocally if $\xi \in A_{R,R^\#}$ deg. 1 primitive then $A_{R,R^\#} \left[\frac{1}{\xi} \right] / (\xi)$ is an untilt of R .

\Rightarrow Untilts of $S \in \text{Def}_{\mathbb{F}_q}$



The construction $\text{Spa}(R, R^\#) \hookrightarrow \text{Spa}(A_{R,R^\#}, A_{R,R^\#}) / V([\xi])$ gives to a construction $S \mapsto Y_S = \mathcal{O}_E$ -adic space



"Untilt of $S =$ deformation of the divisor $\{\mathcal{J}=0\}$ in Y_S to a divisor in $\{\mathcal{J} \neq 0\}$ "

Def $\text{Div}^\diamond: S \in \text{Def}_{\mathbb{F}_q} \quad \text{Div}^\diamond \in \widetilde{\text{Def}}_{\mathbb{F}_q}$

$$\text{Div}^\diamond(S) = \left\{ (\mathcal{L}, \mu) \mid \begin{array}{l} \mathcal{L} \text{ line bundle } / X_S \\ \mu \in H^0(X_S, \mathcal{L}) \end{array} \right\}$$

and s.t. $\forall \bar{S} \rightarrow S$ has deg 1. $\forall S \in S \quad \mu|_{X_{R(S), R(S)^\#}} \neq 0$

The preceding construction defines $(\text{Spa } E)^\diamond \rightarrow \text{Div}^\diamond$
 via $S^\# \hookrightarrow Y_S \rightarrow X_S$ (Cartier div.)
 $(\text{Spa } E)^\diamond / \mathbb{F}_q \xrightarrow{\exists \text{ factorization}} \text{Div}^\diamond$

Proposition - $(\text{Spa } E)^\diamond / \mathbb{F}_q \xrightarrow{\sim} \text{Div}^\diamond$

$\rightarrow \text{Div}^\diamond = \text{q.c. diamond}$

\triangle This is not spatial, not quasi-separated

$$(\text{Spa } E)^\diamond \times_{\text{Div}^\diamond} (\text{Spa } E)^\diamond = \coprod_2 (\text{Spa } E)^\diamond$$

Dis^+
 \downarrow ← representable in proper coh. smooth diamonds
 $\text{Spa } \mathbb{F}_q$) final object in $\widetilde{\text{Ded}} \mathbb{F}_q$
 \uparrow is not quasi-separated

(Prob) $\text{Spa } \mathbb{F}_q((x^{1/n^\infty})) \times_{\text{Spa } \mathbb{F}_q} \text{Spa } \mathbb{F}_q((y^{1/n^\infty}))$
 $= \text{Spa}(\mathbb{F}_q[x^{1/n^\infty}, y^{1/n^\infty}] / V(xy))$